CALIFORNIA INSTITUTE OF TECHNOLOGY

DYNAMICS LABORATORY

GENERAL THEORY OF VIBRATION OF DAMPED LINEAR DYNAMIC SYSTEMS

by

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Introduction

The usual treatment of linearly damped lumped parameter systems assumes that the system equations may be transformed to a symmetrical set of equations. This assumption is justified in passive systems. However, in many problems of interest to aeronautical and electrical engineers the system equations cannot be transformed to a symmetric set of equations. One case in point is the analysis of an aircraft wing under flutter conditions. That non-symmetric systems are physically realizable will be understood when one remembers that it is possible to build any non-symmetric system using an active analog computer. It is the purpose of this report to give a comprehensive analysis of lumped parameter linearly damped second order vibrating systems having symmetric or non-symmetric matrices.

Organization of the Report

The first chapter is a brief resume of useful results from matrix analysis. It states but does not prove theorems. The purpose of the chapter is to explain how matrix analysis may be used to reduce matrices to either diagonal or Jordan form. The second chapter discusses problems solvable in N-space. This class of problems includes the type of problems analyzed by Rayleigh(1), Whittaker(2) and other mathematicians of the late 19th century. At that time the damping matrix was taken to be a linear combination of the inertia and stiffness matrices. However, Caughey(3) showed that the class of problems capable of being solved in

N-space includes those discussed by Rayleigh only as a sub-class. The third chapter takes up those problems which cannot be solved in N-space and the more general analysis of Foss[4], Frazer, Duncan and Collar[5] must be used. As before the type of matrices are not restricted to being symmetric as most earlier work has assumed.

Chapter 4 discusses the very important problem of perturbations of the damping matrix of lumped parameter systems. The main practical use of this type of analysis is in calculating the approximate effect on the frequency and mode shapes of a system when the damping matrix is slightly changed. Finally, Chapter 5 takes up the question of stability of these systems. Unfortunately very little can be said about the stability of second order systems. The usual types of stability tests are not really suitable to the systems analyzed.

At the end of each chapter there is a worked numerical example illustrating the main points of the chapter.

Notation

Capital letters are used for matrices and lower script letters for vectors. Elements of matrices and vectors are in lower script. To avoid unwieldy expressions, except in cases of possible confusion, the brackets about matrices and vectors are often omitted, e.g. $[A]$ may be written as A; $\{\varphi\}$ may be written as φ.

Chapter 1

Resume of Useful Results from Matrix Analysis

As much of the analysis which follows is concerned with the reduction of matrices, a brief resume of useful results from matrix algebra will be presented in the form of definitions and theorems.[(6), (7)]

Definitions: Let the i j element of a matrix $[A]$ be a_{ij}.

Transpose: The transpose of a matrix $[A]$, written $[A]^T$, is obtained by using the rows of A as the columns of $[A]^T$.

Symmetric: A matrix $[A]$ is said to be symmetric if $[A] \equiv [A]^T$.

Orthogonal: A matrix $[A]$ is said to be orthogonal if $[A]^T = [A]^{-1}$ where $[A]^{-1}$ is the inverse of $[A]$, i.e. $[A][A]^{-1} = [1]$.

Similar: Two matrices A and B are said to be similar if P and P^{-1} exist such that $A = PBP^{-1}$.

Eigenvalue: The eigenvalues of a square matrix $[A]$ are the roots of the polynomial $|[\lambda I - A]| = 0$ where $|[\quad]|$ denotes taking the determinant.

Eigenvector: Associated with each eigenvalue λ_i is an eigenvector $\{\varphi_i\}$ which satisfies the following matrix-vector equation

$$[\lambda_i I - A]\{\varphi_i\} = 0$$

Normalization: A vector is said to be normalized if its magnitude is 1.

<u>Theorem 1</u>[(6), (7)] The eigenvalues of real symmetric matrices are real.

<u>Theorem 2</u>[(6), (7)] The eigenvectors associated with distinct eigenvalues of real symmetric matrices are linearly independent and are orthogonal to one another.

Theorem 3[6],[7] Any real symmetric matrix A may be reduced to a diagonal matrix D by an orthogonal transformation φ.

The diagonal elements of D are the eigenvalues of A. The columns of φ are normalized eigenvectors of A. If A has an eigenvalue of multiplicity r there are r linearly independent, though not necessarily orthogonal, eigenvectors associated with this repeated eigenvalue. By the use of the Gram-Schmidt orthogonalization process this set of r linearly independent eigenvectors associated with the repeated eigenvalue, may be reduced to a set of r linearly independent and orthogonal eigenvectors. If x_i $(i = 1, 2, \ldots, n)$ is the original non-orthogonal set and y_i $(i = 1, 2, \ldots, n)$ the desired orthogonal set, the y_i's may be obtained as linear combinations of the x_i's as follows

$$\text{Let} \quad y_1 = x_1$$

$$y_2 = x_2 - \frac{(y_1^T, x_2)}{(y_1^T, y_1)} y_1$$

$$y_3 = x_3 - \frac{(y_2^T, x_3)}{(y_2^T, y_2)} y_2 - \frac{(y_1^T, x_3)}{(y_1^T, y_1)} y_1$$

$$y_4 = x_4 - \frac{(y_3^T, x_4)}{(y_3^T, y_3)} y_3 - \frac{(y_2^T, x_4)}{(y_2^T, y_2)} y_2 - \frac{(y_1^T, x_4)}{(y_1^T, y_1)} y_1$$

where (y_i^T, x_j) is the scalar product of y_i and x_j .

Theorem 4:[6],[7] Any NXN matrix A with a complete set of N linearly independent eigenvectors may be reduced to a diagonal matrix D by a transformation of the type $S^{-1}AS$ where S is a non-singular NXN matrix.

$$[S]^{-1} [A] [S] = [D]$$

The diagonal elements of [D] are the eigenvalues of A. The columns of S are the eigenvectors of A and the rows of $[S]^{-1}$ are the eigenvectors of A^T. It is sometimes useful to evaluate the eigenvectors of the adjoint system A^T and thereby avoid the problem of inverting [S]. However, the system of eigenvectors of A and A^T must now be bio-normalized, i.e. the inner product of $\{\varphi_j\}$ the eigenvector, associated with λ_j, of [A] and $\{q_j\}$ the eigenvector, associated with λ_j, of $[A]^T$ must be 1. As with the symmetric case, eigenvectors associated with repeated eigenvalues need special treatment. Let x_i $(i = 1, 2, \ldots, r)$ be the r linearly independent eigenvectors, associated with the eigenvalue of multiplicity r, of A and y_i $(i = 1, 2, \ldots, r)$ be the corresponding set of linearly independent eigenvectors of A^T. Form a new set y_i^*, each vector of this set being a linear combination of the y_i's.

Let $$y_i^* = \sum_{\ell=1}^{r} a_{i\ell} y_\ell$$

Now $$(y_i^{*T}, x_j) = 0 \qquad i \neq j$$

$$(y_i^{*T}, x_i) = 1$$

$$(y_i^{*T}, x_j) = \sum_{\ell=1}^{r} a_{i\ell}(y_\ell^T, x_j) \qquad \begin{array}{l} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, r \end{array}$$

or

$$I = [A]\,[YX]$$

where the ij^{th} element of A is a_{ij}

where the ij^{th} element of [YX] is (y_i^T, x_j)

$$\therefore \quad A = [YX]^{-1}$$

Theorem 5:[(6), (7)] Given two NXN symmetric matrices A and B with A positive definite there exists a non-singular matrix φ such that

$$[\varphi]^T[A][\varphi] = [1]$$

$$[\varphi]^T[B][\varphi] = D \text{ is a diagonal matrix .}$$

The diagonal elements of D are the roots of the polynomial

$$\|[x[A] - [B]]\| = 0$$

The columns of φ are the eigenvectors of $A^{-1}B$, normalized so that

$$[\varphi]^T[A][\varphi] = [1]$$

To determine $[\varphi]$ knowing that $[\varphi]_1$ is a matrix the columns of which are the eigenvectors of $A^{-1}B$:

$[\varphi]_1[D]$, where $[D]$ is any diagonal matrix, is another matrix the columns of which are eigenvectors of $A^{-1}B$

Let $\qquad \varphi = [\varphi]_1[D]$

$\therefore \qquad [D][\varphi]_1^T[A][\varphi]_1[D] = [1]$

from which $[D]$ may be obtained.

Theorem 6:[(8), (9)] Every square matrix A is reducible to Jordan's canonical form by a transformation of the type

$$\tau^{-1} A \tau = J$$

The Jordan canonical form of a matrix is a quasi diagonal matrix having the eigenvalues of A on the diagonal and the elements immediately above and parallel to the diagonal being either 1 or 0. If the eigenvalues of A are distinct then the Jordan canonical form is strictly diagonal. If *there is an eigenvalue of A of multiplicity r (r > 1) and associated with* this eigenvalue are only $\ell (\ell \le r)$ linearly independent eigenvectors then there are $r - \ell$ of the elements immediately above and parallel to the diagonal equal to 1. The Jordan form in general is as shown below:

$$
\begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & & & & & & & 0 \\
0 & \lambda_2 & 0 & \cdots & & & & & & & 0 \\
0 & \cdot & \lambda_3 & 0 & \cdots & & & & & & 0 \\
0 & \cdots & & \lambda_4 & 0 & \cdots & & & & & 0 \\
 & & & & \ddots & & & & & & \\
0 & \cdots & & & \lambda_i & 1 & 0 & \cdots & & & \\
0 & \cdots & & & & \lambda_i & 1 & 0 & \cdots & & \\
0 & \cdots & & & & & \lambda_i & 1 & 0 & \cdots & \\
0 & \cdots & & & & & & \lambda_i & 0 & \cdots & \\
 & & & & & & & & \ddots & & \\
0 & \cdots & & & & & & & & \lambda_{N-1} & 0 \\
0 & \cdots & & & & & & & & & \lambda_N
\end{bmatrix}
$$

The columns of τ are the eigenvectors and generalized eigenvectors of A arranged as follows: τ is an NXN matrix

$$
\Big[\underbrace{\tau_1\ \tau_2\ \tau_3\ \tau_4}_{\text{Ordinary eigenvectors}}\quad \underbrace{\tau_i^{\,1}\ \tau_i^{\,2}\ \tau_i^{\,3}\ \tau_i^{\,4}}_{\tau_i^{\,1}\text{ is ordinary eigenvector}}\quad \underbrace{\tau_{N-1}\ \tau_N}_{\text{Ordinary eigenvectors}}\Big]
$$

$$
\left.\begin{matrix}\tau_i^{\,2}\\ \tau_i^{\,3}\\ \tau_i^{\,4}\end{matrix}\right\}\ \text{are generalized eigenvectors}
$$

The elements off the diagonal equal to 1 are associated with the generalized eigenvectors.

The generalized eigenvector $\{\varphi\}$ of order k associated with the *eigenvalue* λ_i *of A of multiplicity* $r > k$ *is defined as follows*

$$[\lambda_i I - A]^k \{\varphi_i\} = 0$$

To determine τ first obtain the eigenvalues and ordinary eigenvectors of A. If for any repeated eigenvalue λ_i of multiplicity $\underline{r}$ there are only $\ell (\ell < r)$ linearly independent eigenvectors then the other $(r - \ell)$ generalized eigenvectors must be determined. Find the smallest j, $(j \leq r - \ell)$ such that

$$[\lambda_i I - A]^j = \alpha [\lambda_i I - A]^{j+1}$$

where α is a constant.

Then determine $\{\varphi_i^j\}$ such that

$$[\lambda_i I - A]^j \{\varphi_i^j\} = 0$$

Let $\{\varphi_i^j\} = \{a, b, c, d, \ldots N\}$.

Evaluate the sequence of vectors φ_i^{j-r}, $r = 1, 2, \ldots j-1$

$$\{\varphi_i^{j-1}\} = [\lambda_i I - \lambda] \{\varphi_i^j\}$$

A

$$\{\varphi_i^{j-r}\} = [\lambda_i I - \lambda]^r \{\varphi_i^j\}$$

$$r = 1, 2, \ldots, j-1$$

$\{\varphi_i^1\}$ will be a linear combination of the ordinary eigenvectors of the repeated eigenvalue. The combination constants will be linear combinations of the elements of φ_i^j (i.e., a, b, c ... N). Each φ_i^{j-r} may similarly be decomposed into combinations of linearly independent real vectors. The constants of the combinations are linear combinations of the elements of φ_i^j.

Select in turn sets of values of the elements of $\varphi_i^{\,j}$ so that $\varphi_i^{\,1}$ is equal to a distinct ordinary eigenvector. With each set of values proceed through the j-1 sequence of $\varphi_i^{\,j-r}$ as follows.

Determine the number of distinct vectors which are combined to form $\varphi_i^{\,j-r}$. Note the constants (linear combinations of the elements of $\varphi_i^{\,j}$) multiplying each vector. From the preceding computations some of these constants will be specified. Determine the number of vectors with multiplying constants not completely specified. Select sets of unspecified elements of $\varphi_i^{\,j}$ so that each vector multiplied by an unspecified constant is included in turn with the already specified vectors to form a sequence of

$$\varphi_i^{\,j-r_1}, \; \varphi_i^{\,j-r_2}, \; \varphi_i^{\,j-r_3}, \; \text{etc.}$$

Some of these $\varphi_i^{\,j-r_\ell}$ may be ordinary eigenvectors.

With each set of the unspecified elements and the previous specified ones the process is repeated with $\varphi_i^{\,j-r+1}$ until $\varphi_i^{\,j}$ is reached. At this stage in the computation there exist sequences of ordinary-generalized eigenvectors to form columns of $\boldsymbol{T}$.

See end of chapter for worked example.

<u>Theorem 7</u>:[(8), (9)] Any two NXN matrices A and B are reducible by the same similarity transformation $\boldsymbol{T}$ to at least Jordan form if they commute (AB = BA). Provided A and B communite the following results hold: If A has distinct eigenvalues then both A and B are reducible to diagonal form. For let $\boldsymbol{T}$ be the similarity transformation that reduces A to diagonal form and B to at least Jordan form.

$$AB = BA$$

$$\tau^{-1} A \tau = D$$

$$\tau^{-1} B \tau = J$$

D a diagonal matrix: J a Jordan form matrix.

$$\therefore \quad \tau^{-1} A \tau \tau^{-1} B \tau = \tau^{-1} B \tau \tau^{-1} A \tau$$

$$DJ = JD$$

Taking the ij^{th} element of each matrix

$$D_{ii} J_{ij} = J_{ij} D_{jj}$$

$$\text{as } D_{ii} \neq D_{jj} \qquad J_{ij} = 0 \qquad i \neq j$$

Therefore if A has distinct eigenvalues B is reducible to a diagonal form.

If A has repeated eigenvalues but is still reducible to diagonal form, B will only be reduced to diagonal if it possesses a sufficient number of ordinary eigenvectors. For as above

$$DJ = JD$$

Assume that the 1st α eigenvalues of A are repeated and that the rest are distinct

$$\therefore \quad D_{11} = D_{22} = \qquad\qquad = D_{\alpha\alpha}$$

Taking the ij^{th} element of each matrix

$$i > \alpha \qquad D_{ii} J_{ij} = J_{ij} D_{jj}$$

As $D_{ii} \neq D_{jj}$ $\quad i > \alpha \quad i \neq j$ the lower $n - \alpha$ columns of J must be strictly diagonal.

Again $J_{ij} = 0, \quad j > \alpha \qquad i \neq j$

$$\therefore \quad \tau^{-1} B \tau = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & D \end{array}\right]$$

where X is a $\alpha \times \alpha$ matrix and D is a $n - \alpha$ diagonal matrix.

Premultiply this equation by [R] and postmultiply by $[R]^{-1}$

where

$$[R] = \left[\begin{array}{c|c} [Y] & 0 \\ \hline 0 & [I] \end{array}\right] ; \quad [R]^{-1} = \left[\begin{array}{c|c} [Y]^{-1} & 0 \\ \hline 0 & [I] \end{array}\right]$$

where [Y] is a non-singular $\alpha \times \alpha$ matrix selected below

$$\therefore \quad [R]^{-1}[\tau]^{-1}[B][\tau][R] = \left[\begin{array}{c|c} [Y]^{-1}[X][Y] & 0 \\ \hline 0 & D \end{array}\right]$$

However

$$[R]^{-1} [\tau]^{-1}[A][\tau][R] = [D]$$

and for any [X] there exists a particular similarity transformation [Y] such that

$$[Y]^{-1} [X] [Y]$$

is at least in Jordan form.

From this it is evident that unless [B] has sufficient ordinary eigenvalues $[R]^{-1} [\tau]^{-1}[B] [\tau] [R]$ is in Jordan form. Therefore, if [A] has repeated eigenvalues and [A] and [B] commute but each has a complete set of eigenvectors, then there exists a similarity transformation $[\tau]^* = [\tau][R]$ such that

$$[\tau]^{*-1}[A][\tau]^* = [D_1]$$

$$[\tau]^{*-1}[B][\tau]^* = [D_2]$$

where $[D_1]$ and $[D_2]$ are strictly diagonal.

Moreover if A is reducible only to Jordan form then B must have at least the same number of repeated eigenvalues distributed along the diagonal of the reduced form of [B] in the same manner as on the diagonal of [A]. However [B] may be reducible to a strictly diagonal form.

It should be noted that commutability is a sufficient but not a necessary condition for two matrices to be simultaneously reduced to J form. If the two matrices are reduced to strict diagonal form then commutability is both a necessary and a sufficient condition.

Example:

To illustrate the theorems discussed above the following matrix will be reduced to Jordan form.

$$\text{Let} \qquad [A] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 3-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$= (2 - \lambda \left[(3-\lambda)(1-\lambda) + 1\right]$$

$$\underset{\lambda}{=} (\lambda - 2)^3$$

i.e., $\lambda = 2, 2, 2,$ are the eigenvalues of [A]. To determine the ordinary eigenvectors associated with $\lambda = 2$

$$[A - \lambda I]\{\varphi\} = 0 \qquad\qquad \text{Let } \{\varphi\} = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\therefore \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = 0$$

$$\therefore \quad \begin{aligned} -c &= 0 \\ b + c &= 0 \\ -(b + c) &= 0 \end{aligned}$$

Hence there is only one ordinary eigenvector associated with the repeated eigenvalue of multiplicity 3

$$\{\varphi^I\} = a \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

To determine the generalized eigenvectors associated with the eigenvalue $\lambda = 2$

$$[A - \lambda I]^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A - \lambda I]^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore there is one generalized eigenvector of order 3 and one of order 2.

$$\text{Let} \quad \{\varphi_1^{\,3}\} = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\therefore \quad \{\varphi_1^{\,2}\} = [A - \lambda I]\{\varphi_1^{\,3}\}$$

$$= \begin{Bmatrix} -c \\ b + c \\ -(b + c) \end{Bmatrix}$$

$$= (b + c)\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} - c \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\varphi_1{}^1\} = [A + \lambda I]\{\varphi_1{}^2\}$$

$$= \begin{Bmatrix} b+c \\ 0 \\ 0 \end{Bmatrix} = (b+c)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Let $(b + c) = 1$

$$\{\varphi_1{}^1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\varphi_1{}^2\} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} - c\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Let $c = 1$

$$\{\varphi_1{}^2\} = \begin{Bmatrix} -1 \\ 1 \\ -1 \end{Bmatrix}$$

$$\{\varphi_1{}^3\} = a\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + b\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + c\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} a \\ 0 \\ 1 \end{Bmatrix}$$

Let $a = 1$

$$= \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[\tau]^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore [\tau]^{-1}[A][\tau] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ $[\tau]^{-1}[A][\tau]$ is in Jordan form.

Chapter II

Systems Solvable in N-Space

In this chapter problems solvable in N space will be presented. This set of problems includes as a special class those solved by Rayleigh and Whittaker. In the following analysis the above results of matrix theory will be freely used by merely noting the particular theorem involved.

The equations of motion of lumped parameter linearly damped systems may be written as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{1}$$

where $[M]$ is the mass matrix of order N x N

$[C]$ is the damping matrix of order N x N

$[K]$ is the spring matrix of order N x N

$\{x\}$ is the displacement vector N x 1

dot signifies differentiating w.r.t. time

$\{f(t)\}$ is the forcing function vector of order N x 1

In the most general case there are no restrictions on $[M]$, $[K]$ and $[C]$ as far as symmetry or definiteness of form is concerned. It is possible that all three matrices be singular but for the present it is assumed that M is non-singular.

Before discussing the more general theory a brief resume of the well known theory of symmetric positive semi-definite systems will be presented. In any physically realizable passive (non active) system the matrices must be symmetric and at least non-negative definite.

For active systems there are no such restrictions on the definiteness of the form or on the necessity for symmetry.

1. Existing Theory for Symmetric Semi-definite Systems:

Other requirements for these systems: Either [M] or [K] be positive definite. Here for convenience it is assumed that [M] is non-singular as the case for [M] singular, [K] non-singular follows directly.

Equations of motion

$$M \ddot{x} + C \dot{x} + K x = f(t) \tag{1}$$

To simplify notation the brackets around M, K, C, x and f(t) will not be shown where the context clearly implies matrix notation.

Here M, C, K are symmetric and M is positive definite. By Theorem 5 a N x N matrix Q exists such that

$$\begin{aligned} Q^T M Q &= [1] \\ Q^T K Q &= D_1 \end{aligned} \tag{2}$$

where

Q^T is the transpose of Q

[1] is the identity matrix of order N x N

D_1 is a diagonal matrix of order N x N

If C is such that

$$Q^T C Q = D_2 \tag{2'}$$

is a diagonal matrix then the system specified by Eq. (2) can be solved by the methods outlined by Rayleigh[(1)]. To this end

$$\text{Let} \qquad x = Q \eta \tag{3}$$

where η is a N x 1 column vector the elements of which are functions of time.

∴ on substituting (3) into (2)

$$M Q \ddot{\eta} + C Q \dot{\eta} + K Q \eta = f(t) \tag{4}$$

Premultiply Eq. (4) by Q^T

$$Q^T M Q \ddot{\eta} + Q^T C Q \dot{\eta} + Q^T K Q \eta = Q^T f(t) \tag{5}$$

Now under the assumptions specified

$$Q^T M Q = [I]$$

$$Q^T C Q = D_2$$

$$Q^T K Q = D_1$$

$\therefore$ Eq. (4) may be written as

$$\ddot{\eta} + D_2 \dot{\eta} + D_1 \eta = G(t) \tag{6}$$

where G(t), a N x 1 column vector, is equal to $Q^T f(t)$.

Equation (6) is now in reduced form. In this case all the matrices of Eq. (6) are in diagonal form and so the system of equations may be solved by separating out the individual equations as follows.

The i^{th} equation of Eq. (6) is

$$\ddot{\eta}_i + d_{2i} \dot{\eta}_i + d_{1i} \eta_i = G_i(t) \tag{7}$$

where η_i is the i^{th} element of η

d_{2i} is the i^{th} diagonal element of D_2

d_{1i} is the i^{th} diagonal element of D_1

$G_i(t)$ is the i^{th} element of G(t)

Equation (7) is the well known linearly damped spring mass system under forced excitation $G_i(t)$.

$$\therefore\ \eta_i(t) = e^{-(d_{2i}/2)t}\left[A_i \sin\sqrt{d_{1i}}\sqrt{1 - d_{2i}^2/4d_{1i}}\ t + B_i \cos\sqrt{d_{1i}}\sqrt{1 - \frac{d_{2i}^2}{4d_{1i}}}\ t \right]$$

$$+ \int_0^t G_i(\tau)\, h(t-\tau)\, d\tau \tag{8}$$

where h(t) is response of Eq. (7) to a δ function at t = 0 with zero initial conditions

$$h(t) = \frac{1}{\sqrt{d_{1i} - d_{2i}^2/4}}\ e^{-(d_{2i}/2)\,t} \sin\sqrt{d_{1i} - d_{2i}^2/4}\ t \tag{9}$$

A_i and B_i are arbitrary constants which are determined by the initial conditions as follows:

From Eq. (3)

$$\begin{aligned} x(0) &= Q\eta(0) \\ \dot{x}(0) &= Q\dot{\eta}(0) \end{aligned} \tag{10}$$

but $\eta_i(0) = B_i$

$$\dot{\eta}_i(0) = -\frac{d_{2i}}{2} B_i + \sqrt{d_{1i} - d_{2i}^2/4}\ A_i \tag{11}$$

$\therefore$ From Eq. (10)

$$\eta(0) = Q^{1}\ x(0)$$

$$\eta(0) = Q^{-1}\ x(0)$$

$$\{B_i\} = [Q]^{-1}\{x_i(0)\} \tag{12}$$

and

$$\dot{\eta}(0) = Q^{-1}\ \dot{x}(0)$$

$$\left\{ -\frac{d_{2i}}{2} B_i + \sqrt{d_{1i} - d_{2i}^2/4}\ A_i \right\} = [Q]^{-1}\{\dot{x}(0)\}\ . \tag{13}$$

Knowing $\{x_i(0)\}$ and $\{\dot{x}_i(0)\}$ it is easy to determine $\{A_i\}$ and $\{B_i\}$ from Eq. (12) and (13).

Formally then the solution of linearly damped symmetric physically realizable systems with the damping matrix C satisfying Eq. (2') follows immediately from the application of *Theorem 5*. The only practical difficulty of this method of solution is in the determination of the matrix θ, the columns of which are the eigenvectors of $M^{-1}K$. If one has available a digital computer then such difficulties do not exist as most computer centers have programs available for determining eigenvectors of real matrices. Moreover, for small order systems it is possible to determine the eigenvectors by well known iterative procedures. An example using Stodola's iterative method for the calculation of eigenvectors is presented at the end of the chapter. It is well to note that the eigenvectors in this case are orthogonal in M and K, i.e.

$$\begin{aligned} \{\phi^i\}^T[M]\{\phi^j\} &= 0 \\ & \qquad i \neq j \\ \{\phi^i\}^T[K]\{\phi^j\} &= 0 \qquad i \neq j \end{aligned} \tag{14}$$

where $\qquad \{\phi^i\}$ is the i^{th} column of θ .

Equation (14) follows directly from the assumptions listed under Eq. (2) on application of the formula for matrix multiplication. If A, B and C are 3 N x N matrices the ij element of A B C can be determined as follows:

$$(A\,B\,C)_{ij} \text{ element } = \sum_{\ell} a_{i\ell}\, e_{\ell j}$$

$$\text{where } e_{\ell j} = \sum_{k} \ell_{\ell k}\, e_{kj}$$

$$\text{where } A = [a_{ij}]$$

$$B = [b_{ij}]$$

$$C = [c_{ij}]$$

$$\text{or } (A\,B\,C)_{ij} \text{ element } = \{A_i\}^T B\{C^j\}$$

where $\{A_i\}^T$ is the ith row of A

$\{C^j\}$ is the jth column of C.

The eigenvalues (characteristic values) satisfy equations of the type

$$\lambda_i = -\frac{\phi_i^T K\, \phi_i}{\phi_i^T M\, \phi_i} \le 0 \qquad \text{all } i$$

where λ_i are the eigenvalues of the undamped problem $M\ddot{x} + Kx = 0$.

2. Biorthogonal Systems:

Now systems which do not possess simple orthogonality will be discussed.

Equation of motion

$$M\ddot{x} + C^*\dot{x} + K^*x = f^*(t) \tag{2.1}$$

Here M, C^* and K^* are N x N matrices. M is positive definite and symmetric in most physically realisable (non-relativistic time invarient) systems.

At present the only restriction on Eq. (2.1) is that M be non-singular.

Premultiply Eq. (2.1) by M^{-1}

$$I\ddot{x} + M^{-1}C^{*}\dot{x} + M^{-1}K^{*}x = M^{-1}f^{*}(t) \tag{2.2}$$

I = Identity matrix

Let
$$M^{-1}C^{*} = C$$
$$M^{-1}K^{*} = K$$
$$M^{-1}f^{*}(t) = f(t)$$

Equation (2.2) reduces to

$$I\ddot{x} + C\dot{x} + Kx = f(t) \tag{2.3}$$

The undamped system corresponding to Equation (2.3) is

$$I\ddot{x} + Kx = f(t) \tag{2.4}$$

The adjoint system corresponding to Eq. (2.4) is

$$I\ddot{x} + K^{T}x = f(t) \tag{2.5}$$

where K^T is the transpose of K.

The word adjoint has many meanings in mathematics. Here the adjoint system means the system having as its matrices the transpose of the matrices of the original system.

In point of fact there is little to be gained by working with biorthogonal instead of similar transformations. This is particularly true in the case of repeated roots where the biorthogonal concept has associated numerical difficulties. Even in the case of repeated roots of symmetric matrices the orthogonal transformation is often as difficult to calculate as the inverse of the matrix whose columns are eigenvectors, though these vectors do not necessarily form an orthogonal set. In brief for calculation purposes similarity transformations are often as useful as orthogonal or biorthogonal transformations.

With these comments in mind the following analysis using the concept of the adjoint system is presented.

First solve the homogeneous problem

$$I \ddot{x} + Kx = 0 \tag{2.6}$$

and the adjoint system

$$I \ddot{y} + K^T y = 0 \tag{2.7}$$

If K is such that it has a complete complement of eigenvectors Eq. (2.6) can be solved as follows.

Let Q be a N x N matrix the columns of which are the eigenvectors of K suitably combined (i.e., in the case of repeated eigenvalues) to form a biorthogonal set with the eigenvectors of K^T. Let P be a N x N matrix, the columns of which are the eigenvectors of K^T.

$$\text{Let} \quad x = Q\eta \tag{2.8}$$

From Eq. (2.6)

$$Q\ddot{\eta} + KQ\eta = 0 \tag{2.9}$$

Premultiply Eq.(2.9) by P^T

$$P^T Q \ddot{\eta} + P^T K Q \eta = 0 \tag{2.10}$$

From Theorem 4

$$P^T Q = [1] \tag{2.11}$$

$$P^T K Q = \bar{K} \tag{2.12}$$

where $\bar{K}$ is a diagonal matrix with diagonal elements equal to the eigenvalues of K.

Hence Eq. (2.10) is completely diagonalized and a typical equation is

$$\ddot{\eta}_{ii} + \overline{K}_{ii}\,\eta_i = 0 \qquad (2.13)$$

From which

$$\eta_{ii}(t) = A_i \sin\sqrt{\overline{K}_{ii}}\; t + B_i \cos\sqrt{\overline{K}_{ii}}\; t \qquad (2.14)$$

$$\therefore \qquad \{X\} = [Q]\,\{\eta(t)\}$$

where $\eta(t)$ is a column vector, the i^{th} element of which is

$$A_i \sin\sqrt{\overline{K}_{ii}}\; t + B_i \cos\sqrt{\overline{K}_{ii}}\; t \qquad (2.15)$$

From the initial conditions $\{x(0)\}$ and $\{\dot{x}(0)\}$ A_i, B_i $i = 1, 2 \ldots N$ may be determined.

For $\quad x(0) = [Q]\,\{\eta(0)\} = [Q]\,\{B\}$

where $\{B\}$ is a column vector the i^{th} element of which is B_i .

$$\therefore \qquad \{B\} = [Q]^{-1}\,\{x(0)\}$$

$$\text{Similarly} \quad \{\sqrt{\overline{K}_{ii}}A_i\} = [Q]^{-1}\,\{\dot{x}(0)\}$$

To solve the forced vibration problem

$$I\{\ddot{x}\} + [K]\,\{x\} = \{f(t)\} \qquad (2.16)$$

$$\text{Let } x = [Q]\{\eta\}$$

$$[Q]\,\{\ddot{\eta}\} + [K]\,[Q]\,\{\eta\} = \{f(t)\} \qquad (2.17)$$

Premultiply Eq. (2.17) by P^T where P^T is the matrix the rows of which are eigenvectors of K^T.

$$\therefore \qquad P^T Q\ddot{\eta} + P^T K Q \eta = P^T f(t) = F(t) \qquad (2.18)$$

$$\therefore \qquad I\ddot{\eta} + \overline{K}\eta = F(t) \qquad (2.19)$$

Equation (2.19) is diagonalized and a typical equation is

$$\ddot{\eta}_i + \overline{K}_{ii}\,\eta_i = F_i(t) \tag{2.20}$$

From which

$$\eta_i = A_i \sin\sqrt{\overline{K}_{ii}}\,t + B_i \cos\sqrt{K_{ii}}\,t + \int_0^t \frac{\sin\sqrt{\overline{K}_{ii}}}{\sqrt{K_{ii}}}(t-\tau)\,F_i(\tau)\,d\tau \tag{2.21}$$

As before A_i, B_i $i = 1, 2, \ldots n$ may be determined from initial conditions and so $\{x\}$ may finally be expressed as

$$\{x\} = [Q]\{\eta\}$$

3. Classical Systems.[3]

Consider the system

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = f(t) \tag{3.1}$$

To diagonalize Eq. (3.1) proceed as follows

Let $\{x\} = [R]\{\eta\}$

$$\therefore \quad [M][R]\{\ddot{\eta}\} + [C][R]\{\dot{\eta}\} + [K][R]\{\eta\} = \{f(t)\} \tag{3.2}$$

Premultiply Eq. (3.2) by $[S]$

$$[S][M][R]\{\ddot{\eta}\} + [S][C][R]\,\dot{\eta} + [S][K][R]\{\eta\} = [S]\{f(t)\} \tag{3.3}$$

If Eq. (3.3) is to be completely uncoupled then $[S]$ and $[R]$ must exist such that

$$\begin{aligned} [S][M][R] &= [D_1] \\ [S][C][R] &= [D_2] \\ [S][K][R] &= [D_3] \end{aligned} \tag{3.4}$$

where $[D_1]$, $[D_2]$ and $[D_3]$ are diagonal matrices.

Rearranging the conditions specified by Eq. (3.4)

$$[R]^{-1}[M]^{-1}[S]^{-1}[S][C][R] = [D_1]^{-1}[D_2] \tag{3.5}$$

$$\therefore \quad [R]^{-1}\left([M]^{-1}[C]\right)[R] = [D_4] \tag{3.6}$$

$$[R]^{-1}[M]^{-1}[S]^{-1}[S][K][R] = [D_1]^{-1}[D_3] \tag{3.7}$$

$$[R]^{-1}\left([M]^{-1}[K]\right)[R] = [D_5] \tag{3.8}$$

As $[D_1]^{-1}$ is the inverse of a diagonal matrix it is itself diagonal. The product of 2 diagonal matrices is a diagonal matrix; hence $[D_4]$ and $[D_5]$ are diagonal.

Systems which can be completely diagonalized in N-space are known as classical systems.[3] As systems with singular mass matrices are treated separately for ease of computation classical systems are those systems which satisfy the following conditions:

1. $[M]$ is non-singular
2. There exists a N x N non-singular matrix τ such that

$$\left.\begin{aligned} [\tau]^{-1}\left([M]^{-1}[K]\right)[\tau] &= [D_1] \\ [\tau]^{-1}\left([M]^{-1}[C]\right)[\tau] &= [D_2] \end{aligned}\right\} \tag{3.9}$$

where $[D_1]$ and $[D_2]$ are N x N diagonal matrices.

If any matrix A is diagonalized by a similarity transformation τ then the columns of τ are the eigenvectors of A. Therefore the conditions for a classical system reduce to

1. $[M]^{-1}$ exists
2. $[M]^{-1}[K]$ and $[M]^{-1}[C]$ have the same complete set of eigenvectors.

It should be noted that similarity transformations include orthogonal and biorthogonal transformations as special cases.

In a previous paper Caughey[3] showed that if in the case of symmetric [M] and [K] and distinct eigenvalues of $(M^{-1}K)$

$$[M]^{-1}[C] = \sum_{N=1}^{\infty} \sum_{\ell=0}^{N-1} a_{\ell} \left([M]^{-1}[K]\right)_{1}^{\ell, n} \qquad (3.10)$$

the system will be classical as defined above.

Later proving that fractional powers of matrices may be expressed as linear combinations of the integer powers (using the Caley-Hamilton theorem) he showed that Eq. (3.10) could be further reduced to

$$[M]^{-1}[C] = \sum_{\ell=0}^{N-1} a_{\ell} \left([M]^{-1}[K]\right)^{\ell} \qquad (3.11)$$

When [M] and [K] are general matrices, not necessarily symmetric, a slight extension of Caughey's work shows that if Eq. (3.11) is satisfied and $[M]^{-1}[K]$ is capable of being diagonalized in N space the system is in fact classical. This, however, only demonstrates the sufficiency of Eq. (3.11) for the system to be classical. The necessity of this condition will be demonstrated by proving that for any given $[M]^{-1}[C]$ the a_{ℓ}'s, $\ell = 1, \ldots n$ are uniquely determined.

For a classical system $[M]^{-1}[C]$ must be diagonalizable by the similarity transformation $[\tau]$ which diagonalizes $[M]^{-1}[K]$.

Let $\quad [M]^{-1}[C] = \sum_{\ell=0}^{N-1} a_{\ell} \left([M]^{-1}[K]\right)^{\ell}$

$$\therefore \qquad \tau^{-1}[M]^{-1}[C]\,\tau = \sum_{\ell=0}^{N-1} a_{\ell}\, \tau^{-1} \left([M]^{-1}[K]\right)^{\ell} \tau \qquad (3.12$$

noting that $\quad [\tau]^{-1}\left([M]^{-1}[K]\right)^{\ell}[\tau]$

$$= [\tau]^{-1}\Big(\underbrace{[M]^{-1}[K][\tau][\tau]^{-1}[M]^{-1}[K][\tau][\tau]^{-1}[M]^{-1}[K]\ldots}_{\ell \text{ factors}}\Big)[\tau]$$

$$= [\lambda]^{\ell}$$

where $[\lambda]$ is the diagonal matrix

$$[\lambda] = [\tau]^{-1}\left([M]^{-1}[K]\right)[\tau]$$

Equation 3.12 reduces to

$$[\bar{C}] = \sum_{\ell=0}^{N-1} a_\ell [\lambda]^\ell$$

where $[\bar{C}]$ is a N x N diagonal matrix.

But $[\lambda_i]^\ell = [\lambda_i^{\ell}]$

$$\therefore \{C\} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \ldots\ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 \ldots\ldots & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 \ldots\ldots & \lambda_3^{N-1} \\ \vdots & & & \\ 1 & \lambda_N & \lambda_N^L \ldots\ldots & \lambda_N^{N-1} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \\ a_{N-1} \end{Bmatrix} \tag{3.13}$$

where $\{C\}$ is a vector with elements equal to the diagonal elements of $[\bar{C}]$.

Equation (3.13 may be written as

$$\{C\} = [V]\{a\} \tag{3.14}$$

where $\|[V]\|$ is the well known Vandermonde Determinant

$$\|[V]\|^2 = \left| \prod_{\substack{i,j=1 \\ i \neq j}}^{N} (\lambda_i - \lambda_j) \right| \tag{3.15}$$

where $\| \quad \|$ represents absolute value. From Eq. (3.15) $[V]^{-1}$ exists if $\lambda_i \neq \lambda_j \quad i \neq j$.

From Eq. (3.14)

$$\{a\} = [V]^{-1}\{C\}$$

Hence for distinct eigenvalues of $[M]^{-1}\{K\}$ the condition given by Eq. (3.11) is both necessary and sufficient for the damped system to be classical provided the corresponding undamped system ($[C] \equiv 0$) is classical.

If $[M]^{-1}[K]$ has repeated eigenvalues but a full complement of eigenvectors then the expansion given by Eq. (3.11) is sufficient though not necessary for the system to be classical. The problem is bound up with the minimum polynomial of the matrix $[M]^{-1}[K]$.[7] If a matrix A has distinct eigenvalues then the minimum polynomial of A is simply the polynomial of the characteristic equation. If A has repeated eigenvalues but a full set of ordinary eigenvectors then the minimum polynomial is a product of terms $(\lambda - \lambda_i)$ where each root λ_i, independent of its multiplicity, appears only once. Now every matrix A satisfies its minimum polynomial.

Hence if $[M]^{-1}[K]$ has a repeated eigenvalue λ_1 of multiplicity α and all other roots are distinct but it has a complete set of ordinary eigenvectors.

$$\left([M]^{-1}[K] - \lambda_1 I\right)\left([M]^{-1}[K] - \lambda_{\alpha+1} I\right)\left([M]^{-1}[K] - \lambda_{\alpha+2} I\right)\cdots\left([M]^{-1}[K] - \lambda_N I\right) = 0 \tag{3.16}$$

Equation (3.16) shows that $\left([M]^{-1}[K]\right)^{N-\alpha+1}$ may be expanded in terms of a series of powers of $\left([M]^{-1}[K]\right)$ i.e.

$$[M]^{-1}[K]^{N-\alpha+1} = \sum_{i=0}^{N-\alpha} b_i \, [M]^{-1}[K]^i$$

Thus Caughey's expansion, Eq. (3.11) in the case of repeated eigenvalues reduces to

$$[M]^{-1}[C] = \sum_{\ell=0}^{N-\alpha} a_\ell \left([M]^{-1}[K]\right)^{\ell} \tag{3.17}$$

Due to the fact that the set of eigenvectors associated with the repeated eigenvalues is not unique, for any linear combination of the set is also an eigenvector, $[M]^{-1}[K]$ will be diagonalized by a matrix τ^*.

$$\tau^* = \tau R$$

where τ is any set of eigenvectors of $[M]^{-1}[K]$ and R has the form

$$R = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right]$$

R is an N x N matrix, B is any non-singular $\alpha \times \alpha$ matrix.

Applying the similarity transformation τ^* to Eq. (3.17)

$$\tau^{*-1}[M]^{-1}[C]\,\tau^* = \sum_{\ell=0}^{N-\alpha} a_\ell\, R^{-1}\,\tau^{-1}\left([M]^{-1}[K]\right)^{\ell}\tau R$$

$$\text{or} \quad \{\overline{C}\} = \begin{array}{c} \alpha\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-\alpha} \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-\alpha} \\ \vdots & \vdots & \vdots & & \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-\alpha} \\ 1 & \lambda_{\alpha+1} & \lambda_{\alpha+1} & \cdots & \lambda_{\alpha+1}^{N-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-\alpha} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{N-\alpha} \end{Bmatrix} \tag{3.18}$$

where $\{\overline{C}\}$ is a column vector with elements equal to the diagonal elements of $\tau^*[M]^{-1}[C]\,\tau^*$.

From Eq. (3.18) it is seen that there are only $(N - \alpha + 1)$ independent equations connecting $\{\overline{C}\}$ and $\{a\}$.

Hence if

$\overline{C}_\alpha, \overline{C}_{\alpha+1}, \overline{C}_{\alpha+2} \ldots \overline{C}_N$ are

specified, $a_0, a_1 \ldots a_{N-\alpha}$ are unique.

Also $\overline{C}_1 = \overline{C}_2 = \overline{C}_3 \ldots \overline{C}_\alpha$

Therefore Eq. (3.17) is a sufficient condition for classical systems but it is not necessary as it restricts the class of $[\overline{C}]$. By Eq. (3.17) the lower N-α elements of $[\overline{C}]$ can be arbitrarily specified but the top α elements must be all equal. Therefore to obtain a completely arbitrary $[\overline{C}]$ all that is needed is an additional matrix C_A which when diagonalized by τ has all zeros for the lower N-α elements of the diagonal and specified values for the top α elements of the diagonal. One possible C_A is then C_A^i where

$$\tau^{-1} C_A^i \tau = \begin{array}{cc} \begin{array}{cc} \alpha & N-\alpha \end{array} & \\ \left[\begin{array}{ccccc|c} 0 & & & & & \\ & 0 & & & & \\ & & \overline{C}_i' & & & 0 \\ & & & 0 & & \\ & & & & 0 & \\ \hline & & 0 & & & 0 \end{array}\right] & \begin{array}{c} \alpha \\ \\ N-\alpha \end{array} \end{array}$$

$$\therefore C_A^i = \tau \begin{array}{c} \begin{array}{cc} \alpha & N-\alpha \end{array} \\ \left[\begin{array}{ccccc|c} 0 & & & & & \\ & 0 & & & & \\ & & \overline{C}_i' & & & 0 \\ & & & 0 & & \\ & & & & 0 & \\ \hline & & 0 & & & 0 \end{array}\right] \end{array} \tau^{-1} \tag{3.19}$$

$$= \overline{C}_i' \left(\emptyset_i \rangle \langle q_i^T \right) \tag{3.20}$$

where $\emptyset_i \rangle \langle q_i^T$ is the outer product of $\emptyset_i$, a N x 1 vector which is an eigenvector of $[M]^{-1}[K]$, and q_i^T, a 1 x N vector which is the transpose of an eigenvector of $[K]^T(M^T)^{-1}$. Both $\emptyset_i$ and q_i are associated with the repeated eigenvalue λ_1 and are such that $\emptyset_i q_i^T = 1$.

$\therefore$ a general C_A may be expressed as

$$C_A = \sum_{i=1}^{\alpha} \bar{C}_i{}' \, \phi_i \rangle \langle q_i^T \tag{3.21}$$

[α = multiplicity of repeated root]

Now, however, ϕ_i and q_i are not uniquely specified and to obtain a complete representation for all possible C_A account must be taken that $\tau^* = \tau R$ may be the similarity transformation of the system.

$$\therefore \qquad \phi_i^* = \sum_{\ell=1}^{\alpha} b_{\ell i} \, \phi_\ell \qquad i = 1, 2, \ldots \alpha \tag{3.22}$$

$$q_i^{T*} = \sum_{k=1}^{\alpha} b_{ik}^{-1} \, q_k^T \qquad i = 1, 2, \ldots \alpha \tag{3.23}$$

where $b_{\ell i}$ is the ℓi element of any *non-singular* $\alpha \times \alpha$ matrix $[B]$ and b_{ik}^{-1} is the ik element of $[B]^{-1}$.

Substituting Eq. (3.22) and (3.23) into (3.21) the general expression for C_A is

$$C_A = \sum_{i=1}^{\alpha} \sum_{k=1}^{\alpha} \sum_{\ell=1}^{\alpha} \left[\phi_\ell \rangle \langle q_k^T \right] \left(\bar{C}_i{}' \, b_{\ell i} \, b_{ik}^{-1} \right) \tag{3.24}$$

where ϕ_i, q_i, $i = 1, 2, \ldots \alpha$ is any set of known eigenvectors associated with the repeated root λ_1 of $[M]^{-1}[K]$ and $[K]^T([M]^T)^{-1}$, respectively, $(\bar{C}_i{}' + \bar{C}_\alpha)$, $i - 1, 2, \ldots \alpha$ are the specified first α diagonal elements into which $[M]^{-1}[C]$ is transformed by τ^*, where $[B]$ is the $\alpha \times \alpha$ matrix in $[R]$ connecting τ^* with the known τ.

Therefore if $[M]^{-1}[K]$ has α repeated eigenvalues but a complete set of ordinary eigenvectors the damping matrix C for classical systems must be of the form

$$[C] = [M] \sum_{i=0}^{n-\alpha} a_i \left[[M]^{-1}[K] \right]^i + [M] \sum_{i=1}^{\alpha} \sum_{k=1}^{\alpha} \sum_{\ell=1}^{\alpha} \left[\phi_\ell \rangle \langle q_k^T \right] b_i d_{\ell k} d_{ik}^{-1} \tag{3.25}$$

where ϕ_ℓ, q_ℓ, $\ell = 1, 2, \ldots \alpha$ are the known eigenvectors associated with the repeated root, a_i, b_i are any constants and $d_{\ell k}$ and $d_{\ell k}^{-1}$ *are the* ℓk^{th} elements of $\alpha \times \alpha$ matrices $[D]$ and $[D]^{-1}$.

4. Complete Solution of Classical Problems with Damping.

The equations of motion are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{4.1}$$

Provided $[C]$ can be expanded to Eq. (3.25) and $[M]^{-1}[K]$ has a complete set of eigenvectors the system specified by Eq. (4.1) can always be solved in N-space.

Premultiply Eq. (4.1) by M^{-1}

$$\{\ddot{x}\} + [M]^{-1}[C]\{\dot{x}\} + [M]^{-1}[K]\{x\} = [M]^{-1} f(t) \tag{4.2}$$

Let $[\tau]^{-1}[M]^{-1}[K][\tau] = [D_1]$ a diagonal matrix, then if $[C]$ is specified by (3.25)

$$[\tau]^{-1}\left([M]^{-1}[C]\right)[\tau] = [D_2] \tag{4.3}$$

where $[D_2]$ is a diagonal matrix.

Let $\{x\} = [\tau]\{\eta\}$ (4.4)

where $\{\eta\}$ is a N x 1 column vector

$$\therefore [\tau]\ddot{\eta} + [M]^{-1}[C][\tau]\dot{\eta} + [M]^{-1}[K][\tau]\eta = [M]^{-1}f(t) \tag{4.5}$$

Premultiply Eq. (4.5) by $[\tau]^{-1}$

$$\therefore \quad \{\ddot{\eta}\} + [D_2]\{\dot{\eta}\} + [D_1]\{\eta\} = [\tau]^{-1}[M]^{-1} f(t) \tag{4.6}$$

Eq. (4.6) is a system of uncoupled equations of the type

$$\ddot{\eta}_i + D_{2ii}\,\dot{\eta}_i + D_{1ii}\,\eta_i = F_i(t) \tag{4.7}$$

where $$\{F_i(t)\} = [\tau]^{-1}[M]^{-1}\{f(t)\} \tag{4.8}$$

$$\therefore \eta_i = e^{-D_{2ii}t/2}\left[A_i \sin\sqrt{D_{1ii}}\sqrt{1 - D_{2ii}^2/4D_{1ii}}\;t + B_i \cos\sqrt{D_{1ii}}\sqrt{1-D_{2ii}^2/4D_{1ii}}\right]t$$

$$+ \int_0^t h(r-2)\,F_i(z)\,dz \tag{4.9}$$

where $h(r) = \dfrac{1}{\sqrt{D_{1ii}}\sqrt{1-D_{2ii}^2/4D_{1ii}}} \sin\sqrt{D_{1ii}}\sqrt{1-D_{2ii}^2/4D_{1ii}}\;t$

From which

$$\{x\} = [\tau]\{\eta_i\}$$

The constants A_i and B_i are determined from the initial conditions as shown above.

Non-classical problems: If the system is not classical then the solution can always be obtained by one of the following methods:

1) Reduction of the matrices to Jordan form in N-space.

2) Transforming the problem to 2N space and then reducing the transformed system to either diagonal or Jordan form.

5. Jordan Form in N-space:

If $[M]^{-1}[J]$ is reducible only to Jordan form then the system may still be solvable in N-space if $[M]^{-1}[C]$ is also reducible to at least Jordan form by the same similarity transformation which reduces $[M]^{-1}[K]$. It is well to note that the transformation which reduces $[M]^{-1}[K]$ to Jordan form is not unique for if

$$[\tau]^{-1}\left([M]^{-1}[K]\right)[\tau] = J$$

then $$[\tau^*]^{-1}\left([M]^{-1}[K]\right)[\tau^*] = J$$

if $$[\tau^*] = [\tau][R]$$

where $[R]$ is any non-singular diagonal or Jordan form matrix with the same diagonal pattern as J. If $[R]$ is in Jordan form then it must have exactly the same form as J. If $[M]^{-1}[K]$ is reducible only to Jordan form then there is no series, corresponding to the Caughey series mentioned above, which would insure that the problem can be solved in N-space. In fact, given any matrix $[\bar{C}]$ in Jordan or diagonal form or even in triangular form it is possible to construct a corresponding $[C]$ such that the problem will be solvable in N-space.

For if

$$[\tau]^{-1}[M]^{-1}[K][\tau] = [J]$$

where $[J]$ is a matrix in Jordan form then given $[\bar{C}]$

if $$[M]^{-1}[C] = [\tau^*][\bar{C}][\tau^*]^{-1}$$

where $[\tau^*] = [\tau][R]$; $[R]$ defined as above the system will be solvable in N-space. It should be noted that the diagonal pattern of $[\bar{C}]$ need have no specific relationship with the diagonal pattern of $[J]$.

From Theorem 7 it is known that if $[M]^{-1}[K]$ and $[M]^{-1}[C]$ commute both are reducible by the same transformation. The test of commutibility is only a sufficient condition--it is not a necessary condition for the system to be solvable in N-space. If solution in

N-space is restricted to strictly diagonal forms then the series developed above is both necessary and sufficient. Moreover the test of commutibility is necessary, in this case, though not sufficient.

To solve problems reducible to Jordan form in N-space proceed as follows. For the sake of simplicity the assumption that $[M]^{-1}[K][M]^{-1}[C] = [M]^{-1}[C][M]^{-1}[K]$ is made.

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{5.1}$$

Premultiply Eq. (5.1) by $[M]^{-1}$

$$\{\ddot{x}\} + [M]^{-1}[C]\{\dot{x}\} + [M]^{-1}[K]\{x\} = [M]^{-1}\{f(t)\}$$

Let $\quad [\tau]^{-1}[M]^{-1}[K][\tau] = [J_k] \quad$ a Jordan form matrix

$\quad [\tau]^{-1}[M]^{-1}[C][\tau] = [J_c] \quad$ a Jordan form matrix

The Jordan patterns of J_k and J_c need not be the same but here, for simplicity, it will be assumed that they are the same. This implies that $M^{-1}K$ and $M^{-1}C$ commute.

Let $\quad \{x\} = [\tau]\eta$

$$\therefore \quad [\tau]\{\ddot{\eta}\} + [M]^{-1}[C][\tau]\{\dot{\eta}\} + [M]^{-1}[K][\tau]\{\eta\} = [M]^{-1}\{f(t)\} \tag{5.2}$$

Premultiply Eq. (5.2) by $[\tau]^{-1}$

$$\{\ddot{\eta}\} + [\tau]^{-1}[M]^{-1}[C][\tau]\{\dot{\eta}\} + [\tau]^{-1}[M]^{-1}[K][\tau]\{\eta\} = [\tau]^{-1}[M]^{-1}\{f(t)\}$$

or
$$\{\ddot{\eta}\} + [J_c]\{\dot{\eta}\} + [J_k]\{\eta\} = \{F(t)\} \tag{5.3}$$

where $\quad \{F(t)\} = [\tau]^{-1}[M]^{-1}\{F(t)\}$

Equation (5.3) is a set of equations some of which are completely diagonalized and some coupled. The diagonalized equations may be solved as before as

$$\ddot{\eta} + J_{ii_c}\dot{\eta}_i + J_{ii_k}\eta_i = F_i(t) \tag{5.4}$$

where $\quad J_{ii+1_c} = J_{ii+1_k} = 0$

The coupled equations of type

$$\ddot{\eta}_j + J_{jj_c}\dot{\eta}_j + \dot{\eta}_{j+1} + J_{jj_k}\eta_j + \eta_{j+1} = F_j(t)$$

or

$$\ddot{\eta}_j + J_{jj_c}\dot{\eta}_j + J_{jj_k}\eta_j = F_j(t) - \dot{\eta}_{j+1} - \eta_{j+1} \tag{5.5}$$

may be solved by first solving for η_{j+1} and then substituting into Eq. (5.5). As at least the last one of the N equations of Eq. (5.3) is completely diagonalized this scheme of solution may be started. It should be noted that as η_{j+1} has an oscillatory part the frequency of which is equal to the natural frequency of η_j , the complete solution of η_j given by Eq. (5.5) is unstable (i.e., amplitude increasing with time) if $J_{jj_c} = 0$.

Another type of system which is reducible to Jordan form in N-space is when $[M]^{-1}[K]$ is diagonalizable but $[M]^{-1}[C]$ is only reducible to Jordan form. In this case if $[M]^{-1}[K]$ and $[M]^{-1}[C]$ commute then $[M]^{-1}[K]$ must have at least r repeated eigenvalues where r is the number of repeated eigenvalues of $[M]^{-1}[C]$ associated with generalized eigenvectors. Then $[C]$ as given by Eq.(3.25) is such that $[M]^{-1}[C]$ is diagonalizable in N-space. However, C_A^i may have the form

$$\tau^{-1}C_A^{\,i}\,\tau = \begin{array}{ccc} & \begin{array}{cc}\alpha & \qquad N-\alpha\end{array} & \\ \begin{array}{r}\text{ith row}\ \cdot \\ \alpha \\ \\ N-\alpha\end{array} & \left[\begin{array}{c|c} \begin{matrix}0&&&\\&0&&\\\cdot\ \cdot\ 0&1&&\\&&\ddots&\\&&&0\end{matrix} & 0 \\ \hline 0 & 0\end{array}\right] & \begin{array}{l}\alpha\\ \\ N-\alpha\end{array} \\ & \begin{array}{cc}\alpha & \qquad N-\alpha\end{array} & \end{array} \tag{5.6}$$

$$C_A^{\,i} = \begin{array}{ccc} & \begin{array}{cc}\alpha & \qquad N-\alpha\end{array} & \\ \begin{array}{r}\text{ith row}\ \cdot \\ \alpha \\ \tau \\ N-\alpha\end{array} & \left[\begin{array}{c|c} \begin{matrix}0&&&\\&0&&\\\cdot\ \cdot\ 0&1&&\\&&\ddots&\\&&&0\end{matrix} & 0 \\ \hline 0 & 0\end{array}\right] & \begin{array}{l}\alpha\\ \tau^{-1} \\ N-\alpha\end{array} \\ & \begin{array}{cc}\alpha & \qquad N-\alpha\end{array} & \end{array} \tag{5.7}$$

$$= \qquad \Phi_i \rangle\langle q_{i+1}^{T} \tag{5.8}$$

$$\therefore\ C_A = \sum_{i=1}^{\alpha} \Phi_i \rangle\langle q_{i+1}^{T} \tag{5.9}$$

$$\therefore\ \text{if}\ [C] = [M]\sum_{i=0}^{n-\alpha} a_i\ [M]^{-1}[K]^{\ i} + [M]\sum_{i=1}^{\alpha} \Phi_i \rangle\langle q_{i+1}^{T} \tag{5.10}$$

$[M]^{-1}[C]$ will be reducible to Jordan form. Naturally a $[M]^{-1}[C]$ may be reduced to a Jordan form but with less repeated roots than $[M]^{-1}[K]$ has. In this case $[C]$ may be expanded in a series which is the appropriate combination of Eq.(3.25) and (5.10).

Before concluding this chapter it is well to note that although it is possible to reduce systems to Jordan form in N-space, in practice it is usually more efficient, from the computational viewpoint, to solve these systems in 2N-space.

Example:

Solve the forced vibration problem:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\}$$

Premultiplying by $[M]^{-1}$

$$\{\ddot{x}\} + [M]^{-1}[C]\{\dot{x}\} + [M]^{-1}[K]\{x\} = [M]^{-1}\{f(t)\}$$

$$[M] = I$$

$$[C] = \begin{bmatrix} 9 & 5 & 0 \\ 0 & 4 & 0 \\ -8 & -8 & 1 \end{bmatrix} \qquad f(t) = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \sin t$$

$$[K] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$[C][K] = \begin{bmatrix} 27 & 19 & 0 \\ 0 & 8 & 0 \\ -26 & -26 & 1 \end{bmatrix} = [K][C] \quad ; \quad [M]^{-1} = [M] = I$$

$$[M]^{-1}[K] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\left\| \left[[M]^{-1}[K] - \lambda I \right] \right\| = (3 - \lambda)(2 - \lambda)(1 - \lambda)$$

$\therefore$ $\lambda = 1, 2, 3$ are the eigenvalues of $[M]^{-1}[K]$.

The eigenvectors of $[M]^{-1}[K]$ are

$$\underset{\lambda=1}{\{\phi_1\}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \qquad \underset{\lambda=2}{\{\phi_2\}} = \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \quad ; \quad \underset{\lambda=3}{\{\phi_3\}} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\therefore \quad \tau = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and $$\tau^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\therefore [\tau]^{-1} \left([M]^{-1}[K]\right) [\tau] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

Hence $$[\tau]^{-1}(M^{-1}K)[\tau] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ is diagonal

$$[T]^{-1}[M]^{-1}[C][T] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 5 & 0 \\ 0 & 4 & 0 \\ -8 & -8 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -4 & 9 \\ 0 & 4 & 0 \\ 1 & 0 & -9 \end{bmatrix}$$

$$\therefore [T]^{-1}(M^{-1}C)[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad \text{is diagonal}$$

Let $\{x\} = [T]\{\eta\}$

$$\therefore [T]\{\ddot{\eta}\} + [M]^{-1}[C][T]\{\dot{\eta}\} + [M]^{-1}[K][T]\{\eta\} = [M]^{-1}\{f(t)\}$$

Premultiply by $[T]^{-1}$

$$\therefore [I]\{\ddot{\eta}\} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \{\dot{\eta}\} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \{\eta\} = \begin{Bmatrix} 2 \\ 0 \\ 1 \end{Bmatrix} \sin t$$

Separating out the equations

$$\ddot{\eta}_1 + \dot{\eta}_1 + \eta_1 = 2 \sin t$$

$$\ddot{\eta}_2 + 4\dot{\eta}_2 + 2\eta_2 = 0$$

$$\ddot{\eta}_3 + 9\dot{\eta}_3 + 3\eta_3 = \sin t$$

Assuming zero initial conditions $\eta_1(0) = \eta_2(0) = \eta_3(0) \equiv 0$

$$\dot{\eta}_1(0) = \dot{\eta}_2(0) = \dot{\eta}_3(0) \equiv 0 \quad .$$

$$\therefore \eta_1(t) = -2 \cos t + c^{-0.5t}\{1.160 \sin 0.864t + 2 \cos 0.864t\}$$

$$\eta_2(t) = 0$$

$$\eta_3(t) = 0.024 \sin t - 0.106 \cos t - 0.003\, e^{-17.3t} + 0.109\, e^{-0.7t}$$

$$\{x\} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \left\{ \begin{matrix} -2\cos t + e^{-0.5t}\{1.160 \sin 0.864t + 2\cos 0.864t\} \\ 0 \\ 0.024 \sin t - 0.106 \cos t - 0.003\, e^{-17.3t} + 0.109\, e^{-0.7t} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} 0.024 \sin t - 0.106 \cos t - 0.003\, e^{-17.3t} + 0.109\, e^{-0.7t} \\ 0 \\ -1.894 \cos t - 0.024 \sin t + e^{-0.5t}\{1.160 \sin 0.864t + 2\cos 0.864t\} \\ + 0.003\, e^{-17.3t} - 0.109\, e^{-0.7t} \end{matrix} \right\}$$

Chapter 3

In this chapter multi-degree of freedom systems will be solved by transforming the problem space to 2N space. The transformation used was first developed by Duncan, Collar and Frazer[5] but greatly extended by Foss.[4] However, Foss's work is still too restrictive for the more general problems discussed in this report.

1. Solution in 2N space:

Problems which cannot be solved in N space can always be solved in 2N space provided $[M]^{-1}$ exists.

To the system equations

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = f(t)\} \tag{1.1}$$

add the identity

$$[M]\{\dot{x}\} - [M]\{\dot{x}\} \equiv 0 \tag{1.2}$$

Define

$$\{Z\} = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}$$

On combining Eqs. (1.1) and (1.2)

$$[R]\{\dot{Z}\} + [S]\{Z\} = \{F(t)\} \tag{1.3}$$

where

$$[R] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \quad ; \quad [S] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

$$\{F(t)\} = \begin{Bmatrix} \{0\} \\ f(t) \end{Bmatrix}$$

If [M], [C], and [K] and symmetric [R] and [S] are symmetric, but even if [M] is positive definite [R] is not positive definite. In some cases there exists an orthogonal transformation [T] such that

$$\left.\begin{aligned} [T]'\,[R]\,[T] &= [I] \\ [T]'\,[S]\,[T] &= [D_1] \end{aligned}\right\} \qquad (1.4)$$

where $[D_1]$ is a 2Nx 2N diagonal matrix. Foss used these symmetric matrices and developed orthogonality conditions. Here a more general treatment is given which does not depend on symmetric matrices. The similarity conditions reduce to Foss's orthogonality conditions when the matrices are symmetric.

$$[R]^{-1} = \begin{bmatrix} -[M]^{-1}[C][M]^{-1} & [M]^{-1} \\ [M]^{-1} & [0] \end{bmatrix} \quad \text{provided } [M]^{-1} \text{ exists}$$

$$[R]^{-1}[S] = \begin{bmatrix} +[M]^{-1}[C] & [M]^{-1}[K] \\ -I & [0] \end{bmatrix}$$

Provided $[R]^{-1}[S]$ has sufficient eigenvectors to span the 2N space the original system will be completely diagonalized in 2N space. Even if $[R]^{-1}[S]$ does not possess a complete set of ordinary eigenvectors, the system will be reducible to Jordan form in 2N space.

Solution of problem in 2N space provided M^{-1} exists:

On premultiplying (1.3) by R^{-1}

$$\{\dot{Z}\} + [R]^{-1}[S]\{Z\} = [R]^{-1}\{F(t)\} \qquad (1.5)$$

As any matrix is reducible to its canonical form by a similarity transform let $[\tau]^{-1}[R]^{-1}[S][\tau] = J_s$ (a 2N x 2N matrix in either diagonal or Jordan form).

Let $\{Z\} = [\tau]\{\xi\}$

$$\therefore \quad [\tau]\{\dot{\xi}\} + [R]^{-1}[S][\tau]\{\xi\} = [R]^{-1}\{F(t)\} \tag{1.6}$$

Premultiply by $[\tau]^{-1}$

$$\therefore \quad \{\dot{\xi}\} + [J_s]\{\xi\} = [\tau]^{-1}[R]^{-1}\{F(t)\} = \{G(t)\} \tag{1.7}$$

Equation(1.7) may be solved as previously indicated in the case when the system reduces to Jordan form in N space. It is well to note that due to the peculiar form of Eq. (1.3) any ordinary eigenvector of $[R]^{-1}[\tau]$ has the form

$$\tau_i = \begin{Bmatrix} \alpha_i\{\phi_i\} \\ \\ \{\phi_i\} \end{Bmatrix}$$

where $\{\phi_i\}$ is a Nx 1 vector and α_i is the eigenvalue associated with τ_i. As before there are the usual biorthogonality and orthogonality relationship among the eigenvectors depending on the forms of $[R]$ and $[S]$ and $[R]^{-1}[S]$.

It is interesting to note that a problem which is Jordan in N space will also be Jordan in 2N space and so there is no advantage in going to the 2N space in this case. Whether or not the system possesses sufficient eigenvectors for any repeated root λ_i depends on the rank of the following determinant

$$\left|\left|\left[\lambda_i^2 I + \lambda_i [M]^{-1}[C] + [M]^{-1}[K]\right]\right|\right| \tag{1.8}$$

If the rank of this determinant is greater than $N-\alpha$ where α is the multiplicity of λ_i then the system can only be reduced to Jordan form. See Duncan, Frazer and Collar (5) for a full discussion of this point.

In the 2 Nspace the corresponding determinant is

$$\left\| \left[\lambda_i [R] + [S] \right] \right\|$$

$$\text{or} \left\| \left[\lambda_i I + [R]^{-1}[S] \right] \right\| =$$

$$\left\| \left[\lambda_i I + \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -[I] & [0] \end{bmatrix} \right] \right\| \tag{1.9}$$

Determinant can be further reduced to

$$\begin{array}{c c} & \begin{array}{cc} N \times N & N \times N \end{array} \\ N \times N & \left\| \left[\begin{array}{c|c} \lambda_i I + [M]^{-1}[C] & [M]^{-1}[K] \\ \hline -I & \lambda_i I \end{array} \right]_{N \times N} \right\| \end{array} \tag{1.10}$$

or

$$\lambda_i^{-N} \left\| \left[\begin{array}{c|c} \lambda_i I + [M]^{-1}[C] & \lambda_i^2 I + \lambda_i [M]^{-1}[C] + [M]^{-1}[K] \\ \hline -I & 0 \end{array} \right] \right\| \tag{1.11}$$

provided $\lambda_i \neq 0$

Knowing that the rank of $\left\| \left[[\lambda_i^2 I + \lambda_i [M]^{-1}[C] + [M]^{-1}[K] \right] \right\|$ is greater than $N-\alpha$, it is easy to see by simply rearranging the columns and rows of Eq. (1.11) not involving the $-I$ that the rank of (1.10) is greater than $2N-\alpha$. This shows that if a system is reducible to Jordan form in N space it will still be only reducible to Jordan in 2N space. In fact if a system has $r (r < \alpha)$ generalized eigenvectors in N space it will have $2r$ gneralized eigenvectors in 2N space (r with each value of $\pm\sqrt{\lambda_i^2}$). If in (1.10) $\lambda_i = 0$ it is easy to see that the same result holds.

2. Singular Mass Matrices.

In all the cases discussed above it was assumed that $[M]$ is a non-singular matrix. However, in some systems $[M]$ is singular and in these cases the analysis presented here is useful.

The equation of motion of the system is

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{2.1}$$

where $\|[M]\| = 0$.

In special cases there may exist a τ such that $[\tau]^{-1}[M][\tau]$, $[\tau]^{-1}[C][\tau]$ and $[\tau]^{-1}[K][\tau]$ are all in at least Jordan form. However, such cases are rare and need not be taken up here.

If $[M]$ is singular then there exists a transformation τ such that $[\tau]^{-1}[M][\tau]$ has Jordan form with at least one zero on the diagonal. Whether $[\tau]^{-1}[M][\tau]$ is a strictly diagonal matrix or a matrix in Jordan form there will always be one row of zero's in $[\tau]^{-1}[M][\tau]$ when $\|[M]\| = 0$. Let this row be the i^{th} row.

The system of equations (2.1) may be represented, without loss of generality,

$$[M'] \{\ddot{x}\} + [C']\{\dot{x}\} + [K']\{x\} = \{F(t)\} \tag{2.2}$$

The i^{th} equation is

$$\sum_{j=1}^{N} (C'_{ij}\, \dot{x}_j + K'_{ij}\, x_j) = f'_i(t) \tag{2.3}$$

as $[M']$ is in Jordan form.

where C'_{ij} is the ij element of $[\tau]^{-1}[C][\tau] = [C']$

and K'_{ij} is the ij element of $[\tau]^{-1}[K][\tau] = [K']$

M'_{ij} is the ij element of $[\tau]^{-1}[M][\tau] = [M']$

$f'_i(t)$ is the i^{th} element of $[\tau]^{-1}\{f(t)\} = \{F(t)\}$

Operating on Eq. (2.2) with the operator

$$\left\{C'_{ii}\frac{d}{dt} + K'_{ii}\right\} \tag{2.4}$$

$$C'_{ii}[M']\{\dddot{x}\} + \left[K'_{ii}[M'] + C'_{ii}[C']\right]\{\ddot{x}\} + \left[K'_{ii}[C'] + C'_{ii}[K']\right]\{\dot{x}\}$$

$$+ K'_{ii}[K']\{x\} = C'_{ii}\{\dot{F}(t)\} + K'_{ii}\{F(t)\} \tag{2.5}$$

The i^{th} equation of Eq. (2.5) is

$$C'_{ii}\left(\sum_{j=1}^{N} C'_{ij}\ddot{x}_j + K'_{ij}x_j\right) + K_{ii}\left(\sum_{j=1}^{N} C'_{ij}\dot{x}_j + K'_{ij}x_j\right)$$

$$= C'_{ii}\dot{f}'_i(t) + K'_{ii}f'_i(t) \tag{2.6}$$

But Eq. (2.6) is as would be expected Eq. (2.3) operated on by

$$\left(C'_{ii}\frac{d}{dr} + K'_{ii}\right)$$

The j^{th} equation of Eq. (2.5) is

$$\sum_{k=1}^{N}{}' C'_{ii}M'_{jk}\dddot{x}_k + \sum_{k=1}^{N}{}' \left(K'_{ii}M'_{jk} + C'_{ii}C'_{jk}\right)\ddot{x}_k +$$

$$\sum_{k=1}^{N}{}' \left(K'_{ii}C'_{jk} + C'_{ii}K'_{jk}\right)\dot{x}_k + \sum_{k=1}^{N}{}' K'_{ii}K'_{jk}x_k$$

$$+ C'_{ii}M'_{ji}\dddot{x}_i + K'_{ii}M'_{ji}\ddot{x}_i + C'_{ii}C'_{ji}\ddot{x}_i + K'_{ii}C'_{ji}\dot{x}_i + C'_{ii}K'_{ji}\dot{x}_i$$

$$+ K'_{ii}K'_{ji}x_i = C'_{ii}\dot{f}'_j(t) + K'_{ii}f'_j(t) \tag{2.7}$$

where $\sum_{j=1}^{N}{}' = \sum_{\substack{j=1 \\ j \neq i}}^{N}$

Equation (2.7) may be rewritten as

$$\sum_{k=1}^{N}{}' \left\{ C_{ii}' M_{jk}' \dddot{x}_k + \left(K_{ii}' M_{jk}' + C_{ii}' C_{jk}'\right) \ddot{x}_k \right.$$

$$\left. + \left(K_{ii}' C_{jk}' + C_{ii}' K_{jk}'\right) \dot{x}_k + K_{ii}' K_{jk}' x_k \right\}$$

$$= \left\{ C_{ii}' \frac{d}{dt} + K_{ii}' \right\} \left\{ f_j'(t) - M_{ji}' \ddot{x}_i - C_{ji}' \dot{x}_i - K_{ji}' x_i \right\} \tag{2.8}$$

But from Eq. (2.3)

$$\left(C_{ii}' \frac{d}{dt} + K_{ii}'\right) x_i = f_i'(t) - \left\{ \sum_{k=1}^{N}{}' \left(C_{ik}' \dot{x}_k + K_{ik}' x_k\right) \right\} \tag{2.9}$$

$\therefore$ R.H.S. of (2.8) reduces to

$$\left\{ C_{ii}' \frac{d}{dt} + K_{ii}' \right\} \left(f_j'(t)\right) - M_{ji}' \ddot{f}_i'(t) + \sum_{k=1}^{N}{}' \left\{ M_{ji}' C_{ik}' \dddot{x}_k \right.$$

$$+ M_{ji}' K_{ik}' \ddot{x}_k + C_{ji}' C_{ik}' \ddot{x}_k + C_{ji}' K_{ik}' \dot{x}_k + K_{ji}' C_{ik}' \dot{x}_k$$

$$\left. + K_{ji}' K_{ik}' x_k \right\} - C_{ji}' \dot{f}_i'(t) - K_{ji}' f_i(t) \quad ,$$

or Eq. (2.8) may now be written as

$$\sum_{k=1}^{N}{}' \left\{ C_{ii}' M_{jk}' - M_{ji}' C_{ik}' \right\} \dddot{x}_k + \sum_{k=1}^{N}{}' \left(K_{ii}' M_{jk}' + C_{ii}' C_{jk}' \right.$$

$$\left. - M_{ji}' K_{ik}' - C_{ji}' C_{ik}' \right) \ddot{x}_k + \sum_{k=1}^{N}{}' \left(K_{ii}' C_{jk}' + C_{ii}' K_{jk}' \right.$$

$$\left. - C_{ji}' K_{ik}' - K_{ji}' C_{ik}' \right) \dot{x}_k + \sum_{k=1}^{N} \left(K_{ii}' K_{jk}' - K_{ji}' K_{ik}' \right) x_k$$

$$= \left(C_{ii}' \frac{d}{dt} + K_{ii}' \right) f_j'(t) - \left\{ M_{ji}' \frac{d^2}{dt^2} + C_{ji}' \frac{d}{dt} + K_{ji}' \right\} f_i'(t) \qquad (2.10)$$

$$j = 1, 2, \ldots N \quad \neq i$$

Equation (2.10) may be interpreted as an N-1 degree of freedom 3rd order system

$$[A]\{\dddot{x}\} + [B]\{\ddot{x}\} + [C]\{\dot{x}\} + [D]\{x\} = \{F(t)\} \qquad (2.11)$$

where

$$[A_{jk}] = \left[C_{ii}' M_{jk}' - M_{ji}' C_{ik}' \right]$$

$$[B_{jk}] = \left[K_{ii}' M_{jk}' + C_{ii}' C_{jk}' - M_{ji}' K_{ik}' - C_{ji}' C_{ik}' \right]$$

$$[C_{jk}] = \left[K_{ii}' C_{jk}' + C_{ii}' K_{jk}' - C_{ji}' K_{ik}' - K_{ji}' C_{ik}' \right]$$

$$[D_{jk}] = \left[K_{ii}' K_{jk}' - K_{ji}' K_{ik}' \right]$$

$$\{F_j\} = \left(C_{ii}' \frac{d}{dt} + K_{ii}' \right) f_j'(t) - \left(M_{ji}' \frac{d^2}{dt^2} + C_{ji}' \frac{d}{dt} + K_{ji}' \right) f_i'(t)$$

$$j,k = 1, 2, \ldots N \quad \neq i$$

Here $\|[A_{jk}]\|$ may or may not be zero. If $\|[A_{jk}]\| = 0$, then the same procedure, as shown above, is used to obtain a N-2 set of 4th order equations and so on until the leading matrix is non-singular.

Equation (2.11) may be solved in 3n space as follows:

$$\text{Let}\quad \begin{aligned} \{\ddot{x}\} &= \{p\} = \{\dot{q}\} \\ \{\dot{x}\} &= \{q\} = \{\dot{r}\} \\ \{x\} &= r \end{aligned} \tag{2.12}$$

Equation (2.11) may then be written as

$$\begin{bmatrix} 0 & 0 & A \\ 0 & A & B \\ A & B & C \end{bmatrix} \begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} + \begin{bmatrix} 0 & -A & 0 \\ -A & -B & 0 \\ 0 & 0 & D \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix} \tag{2.13}$$

or

$$[R]\{\dot{Z}\} + [P]\{Z\} = \{G(t)\} \tag{2.14}$$

where $[R]$ and $[P]$ are 3 (N-1) square matrices and are symmetric if A, B, C and D are symmetric.

If $[M]$, $[C]$ and $[K]$ are symmetric in Eq. (2.1) then $[M']$, $[C']$ and $[K']$ are also symmetric and it is easy to see that in this case $[A]$, $[B]$, $[C]$ and $[D]$ are symmetric.

Equation (2.14) may be solved as indicated above provided $[R]^{-1}$ exists

$$\text{where } [R]^{-1} = \left[\begin{array}{c|c|c} A^{-1}\left[(BA^{-1})^2 - CA^{-1}\right] & -A^{-1}BA^{-1} & A^{-1} \\ \hline -A^{-1}BA^{-1} & A^{-1} & 0 \\ \hline A^{-1} & 0 & 0 \end{array}\right]$$

provided A^{-1} exists.

At this point any consistent set of equations of type

$$[A]\ddot{x} + [B]\dot{x} + [C]x = F(t)$$

where $[A]$, $[B]$ and $[C]$ are $N \times N$ matrices may be solved by transforming the problem to $2N$ space and so on to kN space (k an integer) until the leading matrix is non-singular. In the case of $k = 3$ the system is reducible to a set of $3(N-1)$ 1st order equations and $3(N-1)$ initial conditions are needed for the complete solution. $2N-1$ arbirary initial conditions may be specified and the remaining $N-2$ condition obtained from Eq. (2.2). That is

$$\left.\begin{array}{l} x_1(0),\ x_2(0)\ .\ .\ .\ x_i(0)\ .\ .\ .\ x_N(0) \\ \dot{x}_1(0),\ \dot{x}_2(0)\ .\ .\ .\ \qquad\quad .\ .\ .\ x_N(0) \end{array}\right\}$$ but not $\dot{x}_i(0)$ may be

arbitrarily specified. $\dot{x}_i(0)$ is obtained from Eq. (2.3) and $\ddot{x}_1(0)$, $\ddot{x}_2(0)$ . . . $\ddot{x}_N(0)$ but not $\ddot{x}_i(0)$ is obtained from Eq. (2.2). Hence all the necessary initial conditions can be obtained once the $2N-1$ original initial conditions are specified.

For $k > 3$ the situation is more complicated but the essentials are the same as for $k = 3$.

3. Before taking up a discussion of approximate methods of solution of linearly damped lumped parameter systems it is well to note that integral transform techniques may always be used to obtain the solution to any problem in N space. Generally speaking the integral transform approach leads to some computational difficulties, particularly in factorizing the transformed solution into a form suitable for taking inverses. However, it is well to note that in principle all problems may be solved in N space using any of the standard integral transforms. For

convenience the Laplace transform is used here

Definition:

$$L\,f(t) = \int_0^{\infty} e^{-pt} f(t)\,dt = \overline{f}(p) \tag{3.1}$$

$$f(t) = 0 \qquad t < 0$$

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\,\overline{f}(p)\,dp \tag{3.2}$$

where $p = \gamma + i\omega$ is so chosen that Eq. (3.2) exists. Equations (3.1) and (3.2) are known as transform pairs and give the rules for obtaining $\overline{f}(p)$ from f(t) and f(t) from $\overline{f}(p)$. Generally speaking the class of functions of interest in multi-degree of freedom systems possess a Laplace transform. Whereas existence is generally taken for granted,the problem of determining the inverse transform of multi-degree of freedom systems is by no means trivial. The equations of motion in N space may be written

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{3.3}$$

The problem is to solve Eq. (3.3) under the following initial conditions

$$x(t)_{t=0} = x(0)$$

$$\dot{x}(t)\Big|_{t=0} = \dot{x}(0)$$

where $x(0)$ and $\dot{x}(0)$ are given. Taking the Laplace transform of Eq. (3.3)

$$p^2[M]\{\overline{x}(p)\} - p[M]\{\dot{x}(0)\} - p^2[M]\{x(0)\} + p[C]\{\overline{x}(p)\}$$

$$- [C]\{x(0)\} + [K]\{\overline{x}(p)\} = \{\overline{f}(p)\} \tag{3.4}$$

where $\{\overline{x}(p)\}$ is the Laplace transform of x(t)

$\{\overline{f}(p)\}$ is the Laplace transform of f(t)

$\{\bar{f}(p)\}$ can be calculated from f(t)

$\{\bar{X}(p)\}$ as yet unknown

Rearranging terms in Eq. (3.4)

$$\left[p^2[M] + p[C] + [K]\right]\{\bar{X}(p)\} = \{\bar{f}(p)\} + \left[p[M] + [C]\right]\{x(0)\}$$
$$+ p^2[M]\{\dot{x}(0)\} \qquad (3.5)$$

Note that all terms of the R.H.S. of Eq. (3.5) are known

$$\therefore \{\bar{x}(p)\} = \left[p^2[M] + p[C] + [K]\right]^{-1}\left\{\{\bar{f}(p)\} + \left[p[M] + [C]\right]\{x(0)\}\right.$$
$$\left. + p^2[M]\{\dot{x}(0)\}\right\} \qquad (3.6)$$

as [M], [C] and [K] are known, it is possible to calculate $\left[p^2[M] + p[C] + [K]\right]^{-1}$ and therefore to determine $\{\bar{x}(p)\}$ from Eq. (3.6). It should be noted that it is not possible to use a digital computer to determine $\left[p^2[M] + p[C] + K\right]^{-1}$. Knowing $\{\bar{x}(p)\}$ it is possible to calculate x(t), for even if the elements of $\bar{x}(p)$ cannot be factorized readily into suitable fractions, the inverse of which are known, as a last resort Eq. (3.2) connecting a function with its transform may be used. See end of chapter for a worked problem.

4. Simultaneous Triangularization of 2 matrices.

Given the system equations in the form

$$I\ddot{x} + A\dot{x} + Bx = g(t) \qquad (4.1)$$

where in general

$$A = M^{-1}C$$
$$B = M^{-1}K$$
$$g(t) = M^{-1}f(t) \ .$$

If there existed a transformation τ such that

$$\tau^{-1} A \tau = T_1$$

$$\tau^{-1} B \tau = T_2$$

where T_1 and T_2 are triangular matrices the system could be solved in Nspace. It is known from Schur's theorem that any matrix may be reduced by a similarity transformation to triangular form and that such similarity transformation is by no means unique. Unfortunately a survey of the literature on triangularization of matrices revealed only two interesting theorems involving the simultaneous reduction of 2 matrices to triangular form.

(1) In his paper "The Simultaneous Reduction of Two Matrices to Triangle Form" (American Journal of Mathematics, Vol. 37, 1935, pp. 281-293), J. Williamson proves the following theorem:

Let A be a square matrix of order n and let the elementary divisors of $(A - \lambda I)$ be

$$(\lambda - \lambda_1)^{e_1} (\lambda - \lambda_2)^{e_2} \ldots (\lambda - \lambda_t)^{e_t} \quad ; \quad e_1 + e_2 + \ldots e_t = n$$

If A is not derogatory (characteristic polynomial is the minimum polynomial) and if h(A) (AB-BA) is nilpotent (some power of h(A)(AB-BA) = 0, null matrix) for each of the $(e_1 + 1)(e_2 + 1) \ldots (e_t + 1) - r - 1$ polynomials

$$h(A) = (A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \ldots (A - \lambda_t I)^{r_t}$$

$$0 \le r_2 \le e_i \, , \; r_1 + r_2 + \ldots + r_t \le N\text{-}2$$

then there exists a non-singular W, such that $W^{-1} A W$ and $W^{-1} B W$ are in triangle form.

(2) In his book Mirsky gives as an exercise {An Introduction to Linear Algebra Ex. 42, pg. 326} the following theorem.

The matrices of a set G are simultaneously similar to triangular matrices if and only if there exists linearly independent vectors $x_1 \ldots x_n$, such that, for any A in G and any k, $1 \le k \le n$ $A x_k$ is a linear combination of $x_1 \ldots x_k$.

It is easy to see that neither of these two theorems are of much use in the practical computation of the response of systems. The set of B which can be simultaneously triangularized by the set of transformations which triangularizes A is large but does not appear to possess any property which would help in the construction of such a set.

5. Example:

Solution of 2 degree of freedom systems using Laplace transform techniques.

Equation of motion

$$[M]\{\ddot{x}\} + [R]\{\dot{x}\} + [K]\{x\} = \{f(t)\} \tag{5.1}$$

where $[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$[C] = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$

$$[K] = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \tag{5.2}$$

$\{f(t)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ $\{x(0)\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, $\{\dot{x}(0)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

as $[C] = [K] + [M]$ this system is classical and can be readily solved in N space. To solve using Laplace transform

Using Eq. (3.6)

$$\{\overline{x}(p)\} = \Big[p^2[M] + p[C] + [K]\Big]^{-1} \Big\{\overline{f}(p) + \Big[p[M] + [C]\Big]\{x(0)\} + p^2[M]\{\dot{x}(0)\}\Big\} \tag{5.3}$$

On substituting the parameters of the system Eq. (5.2) in Eq. (5.3)

$$\{\overline{x}(p)\} = \begin{bmatrix} p^2 + 4p + 3 & -(p+1) \\ -(p+1) & p^2 + 3p + 2 \end{bmatrix}^{-1} \begin{bmatrix} p+4 & -1 \\ -1 & p+3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{5.4}$$

now $\begin{bmatrix} p^2 + 4p + 3 & -(p+1) \\ -(p+1) & p^2 + 3p + 2) \end{bmatrix}^{-1}$

$$= \frac{1}{\Delta} \begin{bmatrix} (p^2 + 3p + 2) & (p+1) \\ (p+1) & (p^2 + 4p + 3) \end{bmatrix}$$

$$\begin{aligned} \Delta &= (p^2 + 4p + 3)(p^2 + 3p + 2) - (p+1)^2 \\ &= (p+1)^2 \{(p+3)(p+2) - 1\} \\ &= (p+1)^2 \left(p + \frac{5+\sqrt{5}}{2}\right)\left(p + \frac{5-\sqrt{5}}{2}\right) \end{aligned}$$

$$\begin{aligned} \therefore \{\overline{x}(p)\} &= \frac{1}{\Delta} \begin{bmatrix} p^2 + 3p + 2 & p+1 \\ p+1 & p^2 + 4p + 3 \end{bmatrix} \begin{Bmatrix} p+3 \\ p+2 \end{Bmatrix} \\ &= \frac{1}{\Delta} \begin{Bmatrix} (p+2)\{p^2 + 4p + 3 + p + 1\} \\ (p+1)\{p + 3 + p^2 + 5p + 6\} \end{Bmatrix} \\ &= \frac{1}{\Delta} \begin{Bmatrix} (p+2)(p+1)(p+4) \\ (p+1)(p+3)^2 \end{Bmatrix} \end{aligned}$$

$$\{\bar{x}(p)\} = \left\{\begin{matrix}\overline{x_1(p)} \\ \overline{x_2(p)}\end{matrix}\right\}$$

$$\bar{x}_1(p) = \frac{(p+2)(p+4)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \tag{5.5}$$

$$\bar{x}_2(p) = \frac{(p+3)(p+3)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \tag{5.6}$$

To determine $x_1(t)$ and $x_2(t)$ use is made of Eq. (3.2)

$$x(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\, \bar{x}(p) \; dp \tag{5.7}$$

Eq. (5.7) may be evaluated by standard complex integration techniques. For $t > 0$ the contour is closed on the left with a large semi-circle. By Jordan's lemma it is easy to show that

$$\int_r e^{pt} \frac{(p+2)(p+4)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \; dp = 0 \tag{5.8}$$

$$\int_r e^{pt} \frac{(p+3)(p+3)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \; dp = 0 \tag{5.9}$$

where r is the semi-circular part of the contour.

Hence for $t > 0$

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{(p+2)(p+4)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2}} \, dp$$

$$= \frac{1}{2\pi i}\oint_{D} e^{pt} \frac{(p+2)(p+4)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \, dp \qquad (5.10)$$

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{(p+3)(p+3)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \, dp$$

$$= \frac{1}{2\pi i}\oint_{D} e^{pt} \frac{(p+3)(p+3)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \, dp \qquad (5.11)$$

where D is the contour consisting of the line $\gamma - i\infty \rightarrow \gamma + i\infty$ and the large semi-circle on the left.

By Cauchy's theorem

$$\frac{1}{2\pi i}\int_{D} f(p)\, dp = \Sigma \text{ Residues in D}$$

The residue of an analytic function at a singular point p_o is the coefficient of $\frac{1}{1-p_o}$ in the Laurent expansion of the function about p_o.

Hence to calculate $x_1(t)$ and $x_2(t)$ it is necessary to determine the residues of

$$e^{pt}\,\frac{(p+2)(p+4)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \tag{5.12}$$

$$e^{pt}\,\frac{(p+3)(p+3)}{(p+1)(p+\frac{5+\sqrt{5}}{2})(p+\frac{5-\sqrt{5}}{2})} \tag{5.13}$$

(5.12) and (5.13) have poles at $p = -1, \frac{-5+\sqrt{5}}{2}, \frac{-5-\sqrt{5}}{2}$.

Residue of f(p) at a simple pole p_o

$$= \lim_{p\to p_o} (p - p_o)\, f(p)$$

Residues of (5.12)

$$p = -1 \qquad : \quad \frac{12}{4}\, e^{-t} = 3\, e^{-t}$$

$$p = \frac{-5+\sqrt{5}}{2} \qquad : \quad \frac{1+\sqrt{5}}{5-3\sqrt{5}}\, e^{\frac{-5+\sqrt{5}}{2}t}$$

$$p = \frac{-5-\sqrt{5}}{2} \qquad : \quad \frac{1-\sqrt{5}}{5+3\sqrt{5}}\, e^{\frac{-5-\sqrt{5}}{2}t}$$

Residues of (5.13)

$$p = -1 \qquad : \quad 4\, e^{-t}$$

$$p = \frac{-5+\sqrt{5}}{2} \qquad : \quad +\frac{3+\sqrt{5}}{5-3\sqrt{5}}\, e^{\frac{-5+\sqrt{5}}{2}t}$$

$$p = \frac{-5-\sqrt{5}}{2} \qquad : \quad \frac{3-\sqrt{5}}{5+3\sqrt{5}}\, e^{\frac{-5-\sqrt{5}}{2}t}$$

$$\therefore \quad x_1 = 3e^{-t} + \frac{1+\sqrt{5}}{5-3\sqrt{5}}\, e^{\frac{-5+\sqrt{5}}{2}t} + \frac{1-\sqrt{5}}{5+3\sqrt{5}}\, e^{\frac{-5-\sqrt{5}}{2}t}$$

$$x_2 = 4e^{-t} + \frac{3+\sqrt{5}}{5-3\sqrt{5}}\, e^{\frac{-5+\sqrt{5}}{2}t} + \frac{3-\sqrt{5}}{5+3\sqrt{5}}\, e^{\frac{-5-\sqrt{5}}{2}t}$$

$$x_1(0) = x_2(0) = 1$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0$$

Chapter 4

Perturbation Analysis

In this chapter perturbation analysis [11], [12], [13] will be used to estimate the effect of damping on the frequencies and mode shapes of the system. This analysis is particularly useful in systems which cannot be solved in N-space. In many problems of practical interest only the 1st order effects of damping are needed and in these cases the following analysis saves much time and effort. Again if for any reason the damping matrix changes slightly in a system there is no need to recalculate the mode shapes and the frequencies, as perturbation analysis should give sufficient accuracy in most practical cases.

1. Systems which unperturbed have a complete set of orthogonal eigenvectors and all distinct roots.

The equations of the unperturbed system are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{1.1}$$

$[Q]$ exists such that

$$[Q]^T[M][Q] = [D_1]$$

$$[Q]^T[C][Q] = [D_2]$$

$$[Q]^T[K][Q] = [D_3]$$

where $[D_1]$, $[D]$ and $[D_3]$ are diagonal matrices. The solution to Eq. (1.1) is then

$$\{x\} = [Q]\{\eta(t)\}$$

where $\{\eta_i(t)\}$ is the solution to the i^{th} equation of the reduced system of type

$$\bar{M}_i\ddot{\eta}_i + \bar{C}_i\dot{\eta}_i + \bar{K}_i\eta_i = 0$$

If now Eq. (1.1) is perturbed by slightly changing the form of the damping matrix [C] so that

$$[C'] = [C] + \epsilon[\tilde{C}]$$

ϵ a small number

where $[\tilde{C}]$ is such that

$$[Q]^T[\tilde{C}][Q] = [R]$$

where the diagonal terms of [R] are all zero.

The system equations are

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + \epsilon[\tilde{C}]\{\dot{x}\} + [K]\{x\} = 0 \tag{1.2}$$

Let $\bar{\varphi}^n = \varphi^n + \epsilon\Psi^n + \epsilon^2 \theta^n + \ldots$

$\bar{\lambda}_n = \lambda_n + \epsilon\,\mu_n + \epsilon^2\, U_n + \ldots$

where $\bar{\varphi}^n$ is the n^{th} mode of Eq. (1.2)

$\bar{\lambda}_n$ is the n^{th} eigenvalue of Eq. (1.2)

Let $\{x\} = \{\bar{\varphi}^n\}\, e^{\bar{\lambda}_n t}$

$$\therefore\ \bar{\lambda}_n^2[M]\{\bar{\varphi}^n\} + \bar{\lambda}_n[C]\{\bar{\varphi}^n\} + \epsilon\bar{\lambda}_n[\tilde{C}]\{\bar{\varphi}^n\} + [K]\{\bar{\varphi}^n\} = 0 \tag{1.3}$$

Separating out the orders in ϵ from Eq. (1.3)

<u>Zero order</u>: $\lambda_n^2[M]\{\varphi^n\} + \lambda_n[C]\{\varphi^n\} + [K]\{\varphi^n\} = 0$ (1.4)

<u>First order</u>: $\lambda_n^2[M]\{\Psi^n\} + \lambda_n[C]\{\Psi^n\} + [K]\{\Psi^n\} = -\ 2\mu_n\lambda_n[M]\{\varphi^n\}$

$$+\mu_n[C]\{\varphi^n\} + \lambda_n[\tilde{C}]\{\varphi^n\} \tag{1.5}$$

<u>Second order</u>: $\lambda_n^2[M]\{\theta^n\} + \lambda_n[C]\{\theta^n\} + [K]\{\theta^n\} = -\Big((\mu_n^2 + 2\lambda_n\mu_n)[M]\{\varphi^n\}$

$$+\mu_n[C]\{\Psi^n\} + \lambda_n[\tilde{C}]\{\Psi^n\} + U_n[C]\{\varphi^n\} + \mu_n[\tilde{C}]\{\varphi^n\} + 2\lambda_n\mu_n[M]\{\Psi^n\}\Big) \tag{1.6}$$

From the zero th order it is seen that $\{\varphi^n\}$ and λ_n are the n^{th} eigenvector and eigenvalue of the unperturbed system.

The first order perturbations are obtained from Eq. (1.5)

$$\left[\lambda_n^2[M] + \lambda_n[C] + [K]\right]\{\Psi^n\} = -\Big(2\mu_n\lambda_n[M]\{\varphi^n\} + \mu_n[C]\{\varphi^n\} + \lambda_n[\tilde{C}]\{\varphi^n\}\Big) \tag{1.7}$$

As there exists a complete set of ordinary eigenvectors $\{\varphi^j\}$ $j = 1, 2 \ldots N$ to the unperturbed problem $\{\Psi^n\}$ may be expanded as follows:

$$\{\Psi^n\} = \sum_{j=1}^{N} a_{nj}\{\varphi^j\}$$

Premultiply (1.7) by $\{\varphi^\ell\}^T$ and substituting for $\{\Psi^n\}$

$$\therefore \sum_{j=1}^{N} a_{nj}\lambda_n^2\{\varphi^\ell\}^T[M]\{\varphi^j\} + \sum_{j=1}^{N} a_{nj}\lambda_n\{\varphi^\ell\}^T[C]\{\varphi^j\} + \sum_{j=1}^{N} a_{nj}\{\varphi^\ell\}^T[K]\{\varphi^j\}$$

$$= -\Big(2\mu\lambda_n\{\varphi^\ell\}^T[M]\{\varphi^n\} + \mu_n\{\varphi^\ell\}^T[C]\{\varphi^n\} + \lambda_n\{\varphi^\ell\}^T[\tilde{C}]\{\varphi^n\}\Big) \tag{1.8}$$

Remembering that the $\{\varphi^\ell\}$, $\ell = 1, 2, \ldots N$ are an orthogonal set Eq. (1.8) reduces to

$$a_{n\ell}\left[\lambda_n^2\,\overline{M}_{\ell\ell} + \lambda_n\,\overline{C}_{\ell\ell} + \overline{K}_{\ell\ell}\right] = -\lambda_n\{\varphi^\ell\}^T[\tilde{C}]\{\varphi^n\} \tag{1.9}$$

$$\ell, n = 1, 2, \ldots N$$
$$\ell \neq n$$

$$a_{\ell\ell}\left[\lambda_\ell^2\,\overline{M}_{\ell\ell} + \lambda_\ell\,\overline{C}_{\ell\ell} + \overline{K}_{\ell\ell}\right] = -\Big(2\mu_\ell\lambda_\ell\overline{M}_{\ell\ell} + \mu_\ell\overline{C}_{\ell\ell} + \lambda_\ell\{\varphi^\ell\}^T[\tilde{C}]\{\varphi^\ell\}\Big) \tag{1.10}$$

$$\ell = n$$

But by definition

$$\lambda_\ell^2 \; \bar{M}_{\ell\ell} + \lambda_\ell \; \bar{C}_{\ell\ell} + \bar{K}_{\ell\ell} = 0$$

∴ From Eq. (1.10)

$$\mu_\ell = -\frac{\lambda_\ell \{\varphi^\ell\}^T [\tilde{C}] \{\varphi^\ell\}}{2\lambda_\ell M_{\ell\ell} + C_{\ell\ell}} \qquad (1.11)$$

$$\ell = 1, 2 \ldots N$$

From Eq. (1.9)

$$a_{n\ell} = \frac{-\lambda_n \{\varphi^\ell\}^T [\tilde{C}] \{\varphi^n\}}{\left[\lambda_n^2 \bar{M}_{\ell\ell} + \lambda_n \bar{C}_{\ell\ell} + \bar{K}_{\ell\ell}\right]}$$

$$\ell \neq n$$

a_{nn} is obtained from the condition that

$$\{\bar{\varphi}^n\}^T [M] \{\bar{\varphi}^n\} = \bar{M}_{nn} = \{\varphi^n\}^T [M] \{\varphi^n\} + 2\epsilon \, a_{nn} \bar{M}_{nn} + \ldots$$

$$\therefore \quad a_{nn} = 0$$

$$\therefore \Psi^n = \sum_{j=1}^{N}{}' \frac{-\lambda_n \{\varphi^j\}^T [\tilde{C}] \{\varphi^n\}}{\left[\lambda_n^2 \bar{M}_{jj} + \lambda_n \bar{C}_{jj} + \bar{K}_{jj}\right]} \{\varphi^j\} \qquad (1.12)$$

$$\text{where } \sum_{j=1}^{N}{}' = \sum_{\substack{j=1 \\ j \neq n}}^{N}$$

From Eq. (1.11) and (1.12) the corrections to the frequency and mode shapes to 1st order may be written.

Second order perturbations: Eq. (1.5) may be rewritten

$$\left[\lambda_n^2[M] + \lambda_n[C] + [K]\right]\{\theta^n\} = -\Big((\mu_n^2 + 2\lambda_n \mathbf{U}_n)[M]\{\varphi^n\} + \mu_n[C]\{\Psi^n\}$$

$$+ \lambda_n[\tilde{C}]\{\Psi^n\} + \mathbf{U}_n[C]\{\varphi^n\} + \mu_n[\tilde{C}]\{\boldsymbol{\Psi}^n\} + 2\lambda_n\mu_n[M]\{\Psi^n\}\Big) \qquad (1.13)$$

Let $\{\theta^n\} = \sum_{j=1}^{N} \beta_{nj}\, \varphi^j \qquad n = 1, 2 \ldots N$

Premultiply Eq. (1.13) by $\{\varphi^\ell\}^T$ and noting the orthogonality of the vectors $\{\varphi^\ell\}$ $j = 1, 2, \ldots N$.

$$\lambda_n^2 \beta_{n\ell}\, \overline{M}_{\ell\ell} + \lambda_n \beta_{n\ell}\, \overline{C}_{\ell\ell} + \beta_{n\ell}\, \overline{K}_{\ell\ell}$$

$$= -\Big(\mu_n\{\varphi^\ell\}^T[\tilde{C}]\{\varphi^n\} + 2\lambda_n\mu_n a_{n\ell}\overline{M}_{\ell\ell} + \lambda_n\{\varphi^\ell\}[\tilde{C}]\{\Psi^n\}\Big) \qquad (1.14)$$

$$\ell \neq n$$

$$0 = -\Big((\mu_\ell^2 + 2\lambda_\ell\mu_\ell)\, \overline{M}_{\ell\ell} + \mu_\ell \overline{C}_{\ell\ell} + \lambda_\ell\{\varphi^\ell\}[\tilde{C}]\{\Psi^\ell\}$$

$$+ \mu_\ell\{\varphi^\ell\}^T[\tilde{C}]\{\varphi^\ell\} + 2\lambda_\ell\mu_\ell a_{\ell\ell}\overline{M}_{\ell\ell}\Big) \qquad (1.15)$$

$$\ell = n$$

From Eq. (1.14)

$$\beta_{n\ell} = \frac{-\Big(\mu_n\{\varphi^\ell\}^T[C]\{\varphi^n\} + 2\lambda_n\mu_n a_{n\ell}\overline{M}_{\ell\ell} + \lambda_n\{\varphi\}^T[\tilde{C}]\{\Psi^n\}\Big)}{\lambda_n^2\overline{M}_{\ell\ell} + \lambda_n\overline{C}_{\ell\ell} + \overline{K}_{\ell\ell}} \qquad (1.16)$$

$$n, \ell = 1, 2 \ldots N$$

$$n \neq \ell$$

β_{nn} is obtained from the orthogonality condition

$$\{\bar{\varphi}^n\}^T[M]\{\bar{\varphi}^n\} = \overline{M}_{nn}$$

The terms of order ϵ^2 in this orthogonality condition must be zero.

$$2\beta_{nn}\, \overline{M} + \sum_{\ell=1}^{N} a_{n\ell}^2\, \overline{M}_{\ell\ell} = 0$$

$$\beta_{nn} = -\frac{1}{2\overline{M}_{nn}} \sum_{\ell=1}^{n} a_{n\ell}^2 \; \overline{M}_{\ell\ell} \tag{1.17}$$

From Eq. (1.15)

$$U_\ell = -\frac{\mu_\ell^2 \, \overline{M}_{\ell\ell} + \mu_\ell \{\varphi^\ell\}^T [\widetilde{C}]\{\varphi^\ell\} + \lambda_\ell \{\varphi^\ell\}[\widetilde{C}]\{\Psi^\ell\}}{2\lambda_\ell \, \overline{M}_{\ell\ell} + \overline{C}_{\ell\ell}} \tag{1.18}$$

Equations (1.16), (1.17) and (1.18) give the second order corrections to the frequency and the mode shapes of the perturbed system.

From now on in an effort to simplify the algebra the damping matrix will be $\epsilon[C]$, i.e., the unperturbed system is undamped. It will also be assumed that $[M]^{-1}$ exists.

2. Systems which have distinct roots and possess biorthogonality of the eigenvectors in unperturbed condition.

The equations of motion of these systems, provided $[M]^{-1}$ exists may be written

$$I\ddot{x} + \epsilon C\dot{x} + Kx = 0 \tag{2.1}$$

where K is a non-symmetric matrix.

The undamped system ($\epsilon = 0$) possesses a complete set of ordinary eigenvectors.

$$Q K \Phi = D_1 \text{, a diagonal matrix}$$

$$= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \cdot \end{bmatrix} \tag{2.2}$$

$$Q = \Phi^{-1} \tag{2.3}$$

Let $\qquad x = \bar{\varphi}^n \, e^{\bar{\lambda}_n t}$ (2.4)

Substituting Eq. (2.4) into Eq. (2.1)

$$\therefore \left[\bar{\lambda}_n^{\,2} I \bar{\varphi}^n + \epsilon\bar{\lambda}_n \, C \bar{\varphi}^n + K \bar{\varphi}^n\right] e^{\bar{\lambda}_n t} = 0 \tag{2.5}$$

Let
$$\bar{\varphi}_n = \varphi_n + \epsilon \Psi^n + \epsilon^2 \theta^2 \quad + \ldots . \tag{2.6}$$

$$\bar{\lambda}_n = \lambda_n + \epsilon \mu_n + \epsilon^2 U_n^2 \quad + \ldots . \qquad n = 1, 2 \ldots N \tag{2.7}$$

Zeroth order in ϵ :

$$\left[\lambda_n^{\,2} I + [K]\right] \{\varphi^n\} = 0 \tag{2.8}$$

Equation (2.8) is the simple undamped case and therefore $\{\varphi^n\}$ are the eigenvectors of the undamped system.

1^{st} order in ϵ :

$$\left[\lambda_n^{\,2} I + [K]\right] \{\Psi^n\} = -\left(2\lambda_n \mu_n I + \lambda_n [C]\right) \{\varphi^n\} \tag{2.9}$$

Let
$$\Psi^n = \sum_{j=1}^{N} a_{nj} \, \varphi^j$$

as φ^j form a complete set of vectors.

Let τ^{ℓ} be the eigenvector associated with the adjoint system.

Premultiply (2.9) by $\tau^{\ell T}$

$$\therefore \left[\lambda_n^{\,2} \, \tau^{\ell T} + \tau^{\ell T} [K]\right] \{\Psi^n\}$$

$$= -\left(2\lambda_n \mu_n \{\tau^{\ell}\}^T + \lambda_n \{\tau^{\ell}\}^T [C]\right) \{\varphi^n\} \tag{2.10}$$

Now

$$\left[\lambda_{\ell}^{\,2} I + K^T\right] \tau^{\ell} = 0$$

Transposing

$$\tau^{\ell T} \lambda_\ell^2 I + \tau^{\ell T} K = 0 \tag{2.11}$$

Postmultiply by $\{\Psi^n\}$

$$\therefore \left[\lambda_\ell^2 \tau^{\ell T} + \tau^{\ell T}[K]\right]\{\Psi^n\} = 0 \tag{2.12}$$

Subtracting Eq. (2.12) from (2.10)

$$(\lambda_n^2 - \lambda_\ell^2)\{\tau^\ell\}^T\{\Psi^n\} = -\left(2\{\tau^\ell\}^T(\lambda_n\mu_n)\{\varphi^n\} + \{\tau^\ell\}^T(\lambda_n)[C]\{\varphi^n\}\right)$$

$$= -\left(2\lambda_n\mu_n\{\tau^\ell\}^T\{\varphi^n\} + \lambda_n\{\tau^\ell\}^T[C]\{\varphi^n\}\right) \tag{2.13}$$

Now biorthogonality states

$$\{\varphi^n\}^T\{\tau^\ell\} = 0 \qquad n \neq \ell$$

$$= 1 \quad \text{(by normalization)} \quad n = \ell$$

and $\{\tau^\ell\}^T\{\Psi^n\} = a_{n\ell}$

$$\therefore (\lambda_n^2 - \lambda_\ell^2)\, a_{n\ell} = -\lambda_n\{\tau^\ell\}^T[C]\{\varphi^n\}$$

$$n \neq \ell$$

$$a_{n\ell} = -\frac{\lambda_n\{\tau^\ell\}^T[C]\{\varphi^n\}}{\lambda_n^2 - \lambda_\ell^2} \qquad n \neq \ell \tag{2.14}$$

To show that $a_{nn} = 0$ (all n)

By normalizing the eigenvectors of the perturbed system

$$\{\bar{\varphi}^n\}\{\bar{\tau}^n\} = 1$$

As Before $$\bar{\varphi}^n = \varphi^n + \varepsilon\,\Psi^n + \varepsilon^2\,\theta^n$$

$$\bar{\tau}^n = \tau^n + \varepsilon\,R^n + \varepsilon^2 S^n$$

where $\bar{\tau}^n$ is the eigenvector of the perturbed adjoint system.

$$\Psi^n = \sum_{j=1}^{N} a_{nj}\,\varphi^j$$

$$R^n = \sum_{j=1}^{N} b_{nj}\,\tau^j$$

$$\therefore \quad \{\bar{\varphi}^n\}\{\bar{\tau}^n\} = 1 + \varepsilon(a_{nn} + b_{nn}) + 0(\varepsilon^2)$$

as φ^n, τ^n form a biorthonormal set of vectors. From the normalization condition $a_{nn} + b_{nn} = 0$ (all n). Hence a_{nn} and b_{nn} can both be made identically zero without disturbing any of the essential properties of the analysis.

$$\therefore \quad \Psi_n = \sum_{j=1}^{N} -\frac{\lambda_n}{\lambda_n^2 - \lambda_j^2}\left(\{\tau^j\}^T[C]\{\varphi^n\}^T[C]\{\varphi^n\}\right)\{\varphi^j\} \qquad (2.15)$$

$$n = 1, 2, \ldots N$$

In Eq. (2.13) when $n = \ell$

$$-2\lambda_n\mu_n + \{\tau^n\}^T(\lambda_n)[C]\{\varphi^n\} = 0$$

$$\therefore \quad \mu_n = -\frac{1}{2}\{\tau^n\}^T[C]\{\varphi^n\} \qquad n = 1, 2 \ldots N \qquad (2.16)$$

Thus the first order corrections to the perturbed system are given by Eq. (2.15) and (2.16). The second order corrections may be carried out as before.

3. Perturbation Analysis of Symmetric Systems with Repeated Eigenvalues:

The system equations may be written as

$$M \ddot{x} + \epsilon C \dot{x} + K x = 0 \qquad (3.1)$$

where M, K are symmetric, M positive definite and $M^{-1}K$ has a repeated eigenvalue of multiplicity α. Let the repeated root be associated with the α eigenvectors $\phi^i (i = 1, 2, \ldots, \alpha)$ of the unperturbed system. The total set of eigenvectors $\phi^i (i = 1, \ldots \alpha \ldots n)$ form an orthonormal set in M, i.e., $\phi^{nT} M \phi^m = \delta_{nm}$ where δ_{nm} is Kroncker's Delta. Due to the repeated root there may be some difficulty regarding the orthonormalization process. From Theorem (5) we know that if M and K are symmetric and M is positive definite, Q exists such that

$$Q^T M Q = I \qquad (3.1)$$

$$Q^T K Q = \bar{K} \quad \text{(a diagonal matrix)} \qquad (3.2)$$

From (3.1) and (3.2)

$$Q^{-1} M^{-1} (Q^T)^{-1} Q^T K Q = \bar{K}$$

or

$$Q^{-1} (M^{-1} K) Q = \bar{K}$$

Due to the repeated eigenvalue the 1^{st} α diagonal elements of $\bar{K}$ are the same.

Let $\quad Q^* = Q R \qquad (3.4)$

where

$$[R] = \left[\begin{array}{c|c} [L] & C \\ \hline 0 & I \end{array} \right] \qquad (3.5)$$

where $[L]$ is a non-singular $\alpha \times \alpha$ matrix.

From (3.4)

$$(Q^*)^{-1} = R^{-1} Q$$

$$(Q^*)^{-1}(M^{-1}K)\,Q^* = R^{-1}Q^{-1}(M^{-1}K)\,QR$$

From (3.5) $$[R]^{-1} = \left[\begin{array}{c|c} [L]^{-1} & 0 \\ \hline 0 & I \end{array}\right] \tag{3.6}$$

$$\therefore (Q^*)^{-1}(M^{-1}K)\,Q^* = R^{-1}\,\overline{K}\,R \tag{3.7}$$

From the nature of R and R^{-1}

$$R^{-1}\overline{K}R = \left[\begin{array}{c|c} L^{-1}\overline{K}_\alpha L & 0 \\ \hline 0 & \overline{K}_{n-\alpha} \end{array}\right] \tag{3.8}$$

where $\overline{K}_\alpha$ is a $\alpha \times \alpha$ diagonal matrix with diagonal elements equal to the first α diagonal elements of $\overline{K}$, and $\overline{K}_{n-\alpha}$ is a $(n-\alpha) \times (n-\alpha)$ diagonal matrix with diagonal elements equal to the last $(n-\alpha)$ diagonal elements of $\overline{K}$. But due to the repeated eigenvalue the first α diagonal elements of $\overline{K}$ are the same (λ_α say)

$$\therefore R^{-1}\overline{K}R = \overline{K}$$

$$\therefore (Q^*)^{-1}(M^{-1}K)\,Q^* = \overline{K}$$

This means that Q, the similarity transformation matrix for $(M^{-1}K)$, is not unique in this case. However, orthogonality in M restricts the possible class of [L] in [R] above.

From (3.4)

$$Q^{*T} = R^T Q^T \tag{3.9}$$

$$Q^{*T} M Q^* = R^T Q^T M Q R = R^T R \tag{3.10}$$

But $$[R]^T[R] = \left[\begin{array}{c|c} [L]^T[L] & 0 \\ \hline 0 & I \end{array}\right] \tag{3.11}$$

For orthonormalization

$$[L]^T = [L]^{-1} \tag{3.12}$$

To develop the perturbation analysis proceed as follows:

Let $$\overline{\lambda}_n = \lambda_n + \epsilon \mu_n + \epsilon^2 U_n \tag{3.13}$$

$$\overline{\emptyset}_n = \emptyset_n + \epsilon \Psi^n + \epsilon^2 \theta_n \tag{3.14}$$

On substituting Eq. (3.13) and (3.14) into Eq. (3.1) and on separating out the various orders in ϵ

Zeroth order in ϵ:

$$\left[\lambda_n^2 [M] + [K]\right] \{\emptyset^n\} = 0 \tag{3.15}$$

First order in ϵ:

$$\left[\lambda_n^2 [M] + [K]\right]\{\Psi^n\} = -(2\lambda_n \mu_n [M] + \lambda_n [C]\,)\,\emptyset^n \tag{3.16}$$

Let $$\overline{\emptyset^n} = \emptyset^{n*} + \epsilon \Psi^n + \epsilon^2 \theta_n$$

where $\emptyset^{n*}$ is defined as follows:

$$\emptyset^{n*} = \emptyset^n \qquad \alpha + 1 \leq n \leq N$$

$$\emptyset^{n*} = \sum_{j=1}^{\alpha} \alpha nj\, \emptyset^j \qquad 1 < n < \alpha \tag{3.17}$$

Note that (*) does not denote complex conjugate and that αnj is the $j\,n^{th}$ element of the matrix [L] discussed above.

For orthonormality $[\alpha nj]^T = [\alpha nj]^{-1}$ (3.18)

The $\emptyset^{n*}$ are a new set of eigenvectors for the unperturbed system.

From Eq. (3.18)

$$\sum_{i=1}^{\alpha} \alpha^2 ni = 1 \qquad n = 1, 2, \ldots \alpha \tag{3.19}$$

$$\sum_{i=1}^{\alpha} \alpha_{ni}\,\alpha_{mi} = 0 \qquad n \neq m \qquad 1 \le n,\ m \le \alpha \tag{3.20}$$

As before let

$$\Psi^n = \sum a_{nj}\,\phi^{j*} \tag{3.21}$$

First the α_{ij}'s must be determined:

Premultiply Eq. (3.16) by $\phi^{\ell*T}$

$$\therefore \phi^{\ell*T}\left[\lambda_n^2[M] + [K]\right]\Psi^n = -\phi^{\ell*T}\left[2\lambda_n\mu_n[M] + \lambda_n[C]\right]\phi^n \tag{3.22}$$

From the definition of $\phi^{\ell*}$ as an eigenvector of the system

$$\phi^{\ell*T}\left[\lambda_\ell^2[M] + [K]\right]\Psi^n = 0 \tag{3.23}$$

On subtracting (3.23) from (3.22)

$$(\lambda_n^2 - \lambda_\ell^2)\,a_{n\ell} = -\lambda_n\,\phi^{\ell*T}[C]\,\phi^n - 2\lambda_n\mu_n\,\phi^{\ell*T}[M]\,\phi^n \tag{3.25}$$

Let $\quad 1 \le n,\ \ell \le \alpha$

$\therefore \quad \lambda_n = \lambda_\ell = \text{repeated eigenvalue}$

From (3.25) on substituting (3.17)

$$0 = -\lambda_n \sum_{j=1}^{\alpha} \alpha_{\ell j}\,\phi^j[C]\,\phi^n - 2\lambda_n\mu_n\alpha_{\ell n} \tag{3.26}$$

$$\ell, n = 1, 2 \ldots \alpha$$

$$0 = \sum_{j=1}^{\alpha} (\lambda_n\,\phi^{jT}[C]\,\phi^n + 2\lambda_n\mu_n\delta_{nj})\,\alpha_{\ell j} \tag{3.27}$$

$$n, \ell = 1, 2 \ldots \alpha$$

$$= [A]\{L_\ell\}$$

where

$$[A] = \begin{bmatrix} \lambda_n \phi^{1T}[C]\phi^1 + 2\lambda_n\mu_1 & \lambda_n\phi^{2T}[C]\phi^1 \dots \lambda_n\phi^{\alpha T}[C]\phi^1 \\ \lambda_n\phi^{1T}[C]\phi^2 & \lambda_n\phi^{2T}[C]\phi^2 + 2\lambda_n\mu_2 \dots\dots \\ \vdots & \vdots \quad \ddots \quad \vdots \\ \lambda_n\phi^{1T}[C]\phi^\alpha & \dots\lambda_n\phi^{\alpha T}[C]\phi^\alpha + 2\lambda_n\mu_\alpha \end{bmatrix}_\alpha \tag{3.28}$$

and $\{L_\ell\}$ is the ℓ^{th} column of matrix [L].

In (3.27) $\lambda_n = \lambda_1 = \lambda_2 \dots \lambda_\alpha$ the repeated eigenvalue. Now in this form (3.27) can be recognized as a simple eigenvalue problem with $2\lambda_n\mu_i$ $(i = 1, 2 \dots \alpha)$ as the eigenvalues.

Determine the eigenvalues $2\lambda_n\mu_i$ from

$$\|[A]\| = 0 \tag{3.29}$$

Associated with each distinct eigenvalue of [A] is an eigenvector $\{L_i\}$. Provided [A] is such that a complete set of eigenvectors is possible these vectors will be determined uniquely by the conditions of Eq. (3.19) and (3.20)

As before if $\ell > \alpha$

$$a_{n\ell} = - \frac{\lambda_n}{(\lambda_n^2 - \lambda_\ell^2)} \phi^{\ell T}[C]\phi^{n*} \qquad \begin{matrix} \text{all } n \\ \ell > \alpha \end{matrix} \tag{3.30}$$

The values of μ_i $i = \alpha + 1 \dots n$ are determined as in previous case and so μ_i $(i = 1, 2 \dots N)$ is now known.

As in the case treated above with no repeated roots the normalization process implies that

$$\phi^{nT}\,\Psi^n = 0 \qquad \therefore\ a_{nn} \equiv 0 \tag{3.31}$$

$$n = 1, 2 \ldots N$$

To determine $a_{n\ell}$ $\ell, n = 1, 2 \ldots \alpha$ $n \neq \ell$ the second order equation in ϵ is needed.

Second order in ϵ:

$$\left(\lambda_n^2[M] + [K]\right)\{\theta^n\} = -\left[(\mu_n^2 + 2\lambda_n U_n)[M] + \mu_n[C]\right]\phi^{*n}$$

$$- (2\lambda_n\mu_n[M] + \lambda_n[C])\Psi^n \tag{3.32}$$

Premultiply (3.32) by $\phi^{*\ell T}$

$$\phi^{*\ell T}(\lambda_n^2[M] + [K])\,\theta^n = -\,\phi^{*\ell T}\left[(\mu_n^2 + 2\lambda_n U_n)[M] + \mu_n[C]\right]\phi^{*n}$$

$$-\,\phi^{*\ell T}(2\lambda_n\mu_n[M] + \lambda_n[C])\Psi^n \tag{3.33}$$

But from the definition of $\{\phi^{*\ell}\}$

$$\phi^{*\ell T}(\lambda_\ell^2[M] + [K])\,\theta^n = 0 \tag{3.34}$$

On subtracting (3.34) from (3.33)

$$(\lambda_n^2 - \lambda_\ell^2)\,\phi^{*\ell T}[M]\,\theta^n = -\,\phi^{*\ell T}\left[(\mu_n^2 + 2\lambda_n U_n)[M]\right.$$

$$\left. + \mu_n[C]\right]\phi^{*n} - \phi^{*\ell T}\left(2\lambda_n\mu_n[M] + \lambda_n[C]\right)\Psi^n \tag{3.35}$$

if $n, \ell = 1, 2 \ldots \alpha$

Eq. (3.35) reduces to

$$0 = \delta_{\ell n}(\mu_n^2 + 2\lambda_n U_n) + \mu_n\,\phi^{*\ell T}[C]\,\phi^{*n}$$

$$+ 2\lambda_n\mu_n a_{n\ell} + \lambda_n \sum_{j=1}^{N} a_{nj}\,\phi^{*\ell T}[C]\,\phi^{*j} \tag{3.36}$$

$$n, \ell = 1, 2 \ldots \alpha$$

Letting $\ell = n$ in Eq. (3.36)

$$\mu_n^2 + 2\lambda_n U_n + \mu_n \phi^{*nT}[C]\,\phi^{*n} + \lambda_n \sum_{j=1}^{N} a_{nj}\, \phi^{*nT}[C]\,\phi^{*j} = 0 \qquad (3.37)$$

But

$$\lambda_n \sum_{j=1}^{\alpha} a_{nj}\, \phi^{*\ell T}[C]\,\phi^{*j} = \lambda_n \sum_{j=1}^{\alpha} a_{nj} \sum_{s=1}^{\alpha} \sum_{r=1}^{\alpha} \alpha_{\ell s}\, \phi^{s^T}[C]\,\alpha_{jr}\phi^{r}$$

$$= \lambda_n \sum_{j=1}^{\alpha} a_{nj} \sum_{s=1}^{\alpha} \sum_{r=1}^{\alpha} \alpha_{\ell s}\, \alpha_{jr}\, \phi^{s^T}[C]\,\phi^{r} \qquad (3.38)$$

Now from (3.27)

$$\sum_{j=1}^{\alpha} \alpha_{\ell j}\, \lambda_n\, \phi^{j^T}[C]\,\phi^{n} = -2\lambda_n \mu_n\, \alpha_{\ell n} \qquad (3.39)$$

$$\therefore \quad \sum_{r=1}^{\alpha} \alpha_{jr} \sum_{s=1}^{\alpha} \alpha_{\ell s}\, \lambda_n\, \phi^{s^T}[C]\,\phi^{r} = -\sum_{r=1}^{\alpha} 2\alpha_{jr}\lambda_n \mu_r \alpha_{\ell r} \qquad (3.40)$$

On substituting Eq. (3.40) into Eq. (3.36)

$$\delta_{\ell n}(\mu_n^2 + 2\lambda_n U_n) + \mu_n\, \phi^{*\ell T}[C]\,\phi^{*n}$$

$$+ 2\lambda_n \mu_n a_{nj} - \sum_{j=1}^{\alpha} \sum_{r=1}^{\alpha} 2a_{nj}\,\alpha_{jr}\,\lambda_n \mu_r\,\alpha_{\ell r}$$

$$+ \lambda_n \sum_{j=\alpha+1}^{N} a_{nj}\, \phi^{*\ell T}[C]\,\phi^{*j} = 0 \qquad n = 1, 2, \ldots \alpha \qquad (3.41)$$

if $\ell \neq n$ $\quad \delta_{\ell n} = 0$; $\quad a_{nn} = 0 \quad$ all n

Eq. (3.41) may be written as

$$[C][A] + [A][B] = [D] \tag{3.42}$$

where [A], [B], [C] and [D] are $\alpha \times \alpha$ matrices.

$$[A_{ij}] = [a_{ij}]$$

$$[B_{ij}] = \left[\sum_{r=1}^{\alpha} 2\lambda_n \alpha_{ir} \mu_r \alpha_{jr}\right]$$

$$[C_{ij}] = \left[2\lambda_n \mu_i \delta_{ij}\right]$$

$$[D_{ij}] = -\Big[(\mu_i^2 + 2\lambda_n U_i)\delta_{ij} + \mu_i \, \phi^{*jT}[C]\,\phi^{*i}$$

$$+ \lambda_n \sum_{r=\alpha+1}^{N} \dot{a}_{ir} \, \phi^{*jT}[C]\,\phi^{*r}\Big]$$

Equation (3.42) is a well known matrix equation and may be solved by methods presented by Bellman[6]. In this way the various orders in ϵ may be calculated and so the perturbed problem may be solved to any desired accuracy.

4. Perturbation Analysis of Systems with Repeated Eigenvalues but with Sufficient Eigenvectors in Biorthogonal Cases:

The equations of motion of the system may be written as

$$I\ddot{x} + \epsilon C\dot{x} + Kx = 0 \tag{4.1}$$

Let $Q\Phi = I$ $\qquad Q = [\Phi]^{-1}$ (4.2)

$QK\Phi = \bar{K}$ a diagonal matrix with the first α diagonal elements equal.

From (4.2)

$$[\Phi]^T \ K^T \ [\Phi]^{T^{-1}} = [\bar{K}] \tag{4.3}$$

As before let $\quad X = e^{\bar{\lambda}_n t}\{\bar{\phi}_n\}$ (4.4

$$\begin{aligned} \bar{\lambda}_n &= \lambda_n + \epsilon\,\mu_n + \epsilon^2 U_n \ \ldots \\ \bar{\phi}_n &= \phi_n + \epsilon\,\Psi^n + \epsilon^2\,\theta^n \ \ldots \qquad n = 1, 2, \ldots N \end{aligned} \tag{4.5}$$

On substituting (4.5) into (4.1) and separating out the various orders in ϵ:

Zeroth order in ϵ:

$$(\lambda_n^2 I + [K])\,\phi^n = 0 \tag{4.6}$$

First order in ϵ:

$$(\lambda_n^2 I + [K])\,\Psi^n = -\left(2I\,\lambda_n \mu_n + \lambda_n [C]\right)\phi^n \tag{4.7}$$

As before let

$$\bar{\phi}^n = \phi_n^* + \epsilon\,\Psi^n + \epsilon^2\,\theta^n + \ldots \tag{4.8}$$

$$\phi_n^* = \sum_{j=1}^{\alpha} \alpha_{nj}\,\phi^j \qquad n = 1, 2 \ldots \alpha \tag{4.9}$$

$$= \phi_n \qquad n = \alpha + 1, \ldots N$$

Let τ^{ℓ} be an eigenvector associated with the adjoint system K^T. Let τ^{ℓ} ($\ell = 1, 2 \ldots n$) and ϕ^{ℓ} ($\ell = 1, 2 \ldots n$) form a biorthonomal system of eigenvectors.

Premultiply Eq. (4.7) by $\tau^{\ell T}$

$$\tau^{\ell^T}(\lambda_n^2 I + [K])\Psi^n = -\tau^{\ell^T}(2I\lambda_n\mu_n + \lambda_n[C])\,\phi^{n*} \tag{4.10}$$

ϕ^{n*} replaces ϕ^n as defined in Eq. (4.9). From the definition of τ^{ℓ} as an eigenvector of the adjoint system

$$\{\tau^{\ell}\}^T\left[\lambda_{\ell}^2 I + [K]\right]\{\Psi^n\} = 0 \tag{4.11}$$

On subtracting (4.11) from (4.10)

$$(\lambda_n^2 - \lambda_{\ell}^2)\,\tau^{\ell^T}\Psi^n = -\tau^{\ell^T}\left(2\lambda_n\mu_n I + \lambda_n[C]\right)\phi^{n*} \tag{4.12}$$

If n, $\ell = 1, 2, \ldots \alpha$ then Eq. (4.12) reduces to

$$\tau^{\ell^T}\left(2\lambda_n\mu_n I + \lambda_n[C]\right)\phi^{n*} = 0 \tag{4.13}$$

On substituting Eq. (4.9) in Eq. (4.13)

$$2\lambda_n\mu_n\alpha_{n\ell} + \sum_{j=1}^{\alpha} \alpha_{nj}\lambda_n\tau^{\ell^T}[C]\,\phi^j = 0 \tag{4.14}$$

Eq. (4.14) may be rearranged as

$$\sum_{j=1}^{\alpha}\left[\lambda_n\tau^{\ell^T}[C]\,\phi^j + 2\lambda_n\mu_n\delta_{\ell j}\right]\alpha_{nj} = 0 \tag{4.15}$$

$$n, \ell = 1, 2 \ldots \alpha$$

Eq. (4.15) is in the form of the standard matrix equation

$$[A][B] + [C][A] = 0$$

where [A], [B] and [C] are $\alpha \times \alpha$ matrices

$$[A_{ij}] = \alpha_{ij}$$

$$[B_{ij}] = \left[\lambda_n\,\tau^{j^T}[C]\,\phi^i\right]$$

$$[C_{ij}] = \left[2\lambda_n\,\mu_i\,\delta_{ij}\right]$$

and as before(6) α_{nj}($n, j = 1, 2 \ldots \alpha$) and μ_n may be determined. To determine the α_{nj}'s uniquely it is necessary to consider normalization. The new eigenvector matrix $[\Phi^*]$ is a linear transformation of the original eigenvector matrix $[\Phi]$

$$[\Phi^*] = [\Phi][R] \tag{4.16}$$

where

$$[R] = \begin{array}{c|cc|c} & \alpha & N-\alpha & \\ \alpha & [L] & 0 & \alpha \\ & 0 & I & N-\alpha \\ & \alpha & N-\alpha & \end{array} \tag{4.17}$$

where $[L] = [\alpha_{ij}]$ is an $\alpha \times \alpha$ matrix.

Now $[Q]$ is eigenvector matrix of the adjoint system and by definition

$$[Q]^T [\Phi] = I \tag{4.18}$$

Let the new eigenvector matrix of the adjoint system be

$$[Q^*] = [Q][S]$$

$$\therefore \qquad [Q^*]^T = [S]^T [Q]^T \tag{4.19}$$

$$[Q^*]^T [\Phi^*] = [S]^T [Q]^T [\Phi][R]$$

$$= [S]^T [R]$$

$$\therefore \quad \text{For biorthonormality } [S]^T = [R]^{-1} \tag{4.20}$$

From Eq. (4.20) it is seen that beyond the requirement that R is non-singular there is no difficulty about normalization.

To determine the first order corrections in the eigenvectors

$$\text{Let} \qquad \Psi^n = \sum_{j=1}^{N} a_{nj}\, \phi^{j*} \tag{4.21}$$

Substituting Eq. (4.21) into Eq. (4.12)

$$(\lambda_n^2 - \lambda_\ell^2)\, \tau^{\ell * T} \sum_{j=1}^{N} a_{nj}\, \phi^{j*} = -\tau^{\ell * T}\Big(2 I \lambda_n \mu_n + \lambda_n [C]\Big)\, \phi^{n*} \tag{4.22}$$

Note that $\tau^{\ell * T}$ has replaced $\tau^{\ell T}$ where $\tau^{\ell *}$ is defined from Eq. (4.19)

From (4.22) and (4.20)

$$(\lambda_n^2 - \lambda_\ell^2)\, a_{n\ell} = -2\lambda_n \mu_n \delta_{\ell n} - \lambda_n \tau^{\ell * T} [C]\, \phi^{n*} \tag{4.23}$$

If $n \neq \ell$

$\ell > \alpha \qquad n = 1, 2, \ldots \alpha$

$$a_{n\ell} = -\frac{\lambda_n}{(\lambda_n^2 - \lambda_\ell^2)}\, \tau^{\ell * T} [C]\, \phi^{n*} \tag{4.24}$$

If $n \neq \ell$

$m > \alpha \quad \ell = 1, 2 \ldots \alpha$

$$a_{n\ell} = - \frac{\lambda_n}{(\lambda_n^2 - \lambda_\ell^2)} \tau^{\ell * T} [C] \phi^{n*} \tag{4.25}$$

To determine $a_{n\ell} \quad \ell, n = 1, 2 \ldots \alpha \quad n \neq \ell$

the second order approximation is needed. As in the biorthogonal case with a distinct set of eigenvalues $a_{nn} = 0$ all n by consideration of normalization.

Second order in ϵ:

$$\left(\lambda_n^2 I + [K]\right)\{\theta^n\} = - \left[(\mu_n^2 + 2\lambda_n U_n) \left[I + \mu_n [C] \right] \right] \phi^{n*}$$

$$- \left[2\lambda_n \mu_n [I] + \lambda_n [C] \right] \psi^n \tag{4.26}$$

Premultiply by $\tau^{\ell * T}$

$$\tau^{\ell * T} \left[\lambda_n^2 I + [K] \right] \theta^n = - \tau^{\ell * T} \left[(\mu_n^2 + 2\lambda_n U_n) \right.$$

$$\left. \left[I + \mu_n [C] \right] \right] \phi^{n*} - \tau^{\ell * T} \left[2\lambda_n \mu_n [I] + \lambda_n [C] \right] \psi^n \tag{4.27}$$

From definition of $\tau^{\ell *}$

$$\tau^{\ell * T} \left[\lambda_\ell^2 I + [K] \right] \theta^n = 0 \tag{4.28}$$

Subtracting (4.28) from (4.27)

$$(\lambda_n^2 - \lambda_\ell^2) \tau^{\ell * T} \theta^n = - (\mu_n^2 + 2\lambda_n U_n) \delta_{n\ell}$$

$$- \mu_n \tau^{\ell * T} [C] \phi^{n*} - \tau^{\ell * T} \left[2\lambda_n \mu_n [I] + \lambda_n [C] \right] \psi^n \tag{4.29}$$

If $n, \ell = 1, 2 \ldots \alpha \quad \lambda_n = \lambda_\ell$

Equation (4.29) reduces to

$$(\mu_n^2 + 2\,\lambda_n U_n)\, a_{n\ell} + 2\lambda_n \mu_n\, a_{n\ell} + \mu_n \tau^{\ell *^T} [C]\, \phi^{n*}$$

$$+ \lambda_n\, \tau^{\ell *^T} [C]\, \Psi^n = 0 \tag{4.30}$$

$$n, \ell = 1, 2, \ldots \alpha$$

Eq. (4.30) is similar to Eq. (3.36) of Section 3 of this chapter. Proceeding in the same way as was indicated there $a_{n\ell}$ $(n, \ell = 1 \ldots \alpha)$ and U_n, $n = 1, 2 \ldots \alpha$ may be determined.

At this point the power of perturbation analysis should be noted. Once the general formulae have been determined the various corrections require very routine type calculations which can easily be done on a desk calculator. It should be noted that systems which are in Jordan's Form when unperturbed may be capable of complete diagonalization after a suitable perturbation. For this reason perturbation analysis of these systems is not possible by ordinary techniques as the resulting system after the perturbation may or may not be in Jordan's form.

<u>Example</u>[11]. To illustrate the results of the above analysis, consider the following system:

$$[M]\{\ddot{x}\} + [C']\{\dot{x}\} + [K]\{x\} = 0 \ , \tag{1}$$

where

$$\left.\begin{aligned} [M] &= I \\ [K] &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ [C'] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \varepsilon &= 0.1 . \end{aligned}\right\} \tag{2}$$

Undamped System. For the undamped system,

$$\{\phi^1\} = \frac{1}{2}\begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}, \quad \omega_1 = 0.765366 \ ;$$

$$\{\phi^2\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}, \quad \omega_2 = 1.414214 \ ; \tag{3}$$

$$\{\phi^3\} = \frac{1}{2}\begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}, \quad \omega_3 = 1.847759 \ .$$

With use of perturbation scheme of this chapter, the damped natural frequencies are

$$\begin{aligned} \omega_{1d} &\simeq 0.765687 > \omega_1 \ , \\ \omega_{2d} &\simeq 1.413993 < \omega_2 \ , \\ \omega_{3d} &\simeq 1.846696 < \omega_3 \ . \end{aligned} \tag{4}$$

The exact values obtained by solving Eq. (1) are

$$\omega_{1d} = 0.765688 ,$$

$$\omega_{2d} = 1.413990 , \tag{5}$$

$$\omega_{3d} = 1.846698 .$$

Comparison of (4) and (5) shows excellent numerical agreement. It should be noted that the damped natural frequency of the first mode is higher than that for the undamped system, while the damped frequencies for the second and third modes are lower than the corresponding values for the undamped system.

Chapter 5

Stability

The problem of the stability of dynamical systems is an important one and to a large extent has not yet been satisfactorily resolved.

Provided $[M]^{-1}$ exists the equation of motion of a system may be written in 2N space as

$$I\{\dot{Z}\} + \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -[1] & [0] \end{bmatrix} \{Z\} = \{G(t)\} \qquad (1)$$

where $[M]$, $[K]$, and $[C]$ are the mass, spring and damping matrices, respectively.

Solving the homogeneous part of Eq. (1) by letting

$$\{Z\} = \{\varphi_i\} e^{\lambda_i t} \qquad (2)$$

$$\left[\lambda_i [I] + \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -[1] & [0] \end{bmatrix} \right] \{\varphi_i\} = 0 \qquad (3)$$

The values of λ_i which satisfy (3) are given by

$$\left\| \left[\lambda_i I + \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -[1] & [0] \end{bmatrix} \right] \right\| = 0$$

or

$$\left\| \begin{matrix} \left[[M]^{-1}[C] + \lambda_i I \right] & [M]^{-1}[K] \\ -[1] & \lambda_i [I] \end{matrix} \right\| = 0$$

$$\text{or} \qquad \left\| \begin{bmatrix} \lambda_i^2 I + \lambda_i [M]^{-1}[C] + [M]^{-1}[K] & [M]^{-1}[K] \\ 0 & \lambda_i I \end{bmatrix} \right\| = 0$$

$$\left\| \left[\lambda_i^2 I + \lambda_i [M]^{-1}[C] + [M]^{-1}[K] \right] \right\| = 0$$

$$\left\| \left[\lambda_i^2 [M] + \lambda_i [C] + [K] \right] \right\| = 0 \tag{4}$$

Hence the roots of the system in 2N space are identical to those of the N space problem. However, it is easier to discuss the stability in the 2N space.

Taking the homogeneous problem in 2N space

$$I\{\dot{Z}\} + \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -[1] & 0 \end{bmatrix} \{Z\} = 0$$

$$I\{\dot{Z}\} + [R]\{Z\} = 0 \tag{5}$$

$$[R] = \begin{bmatrix} [M]^{-1}[C] & [M]^{-1}[K] \\ -1 & [0] \end{bmatrix}$$

$[R]$ can be reduced to at least Jordan form

Let $[\tau]^{-1}[R][\tau] = [J]$

$\therefore$ if $\{Z\} = [\tau]\eta$

$$I\{\dot{\eta}\} + [J]\{\eta\} = 0 \tag{6}$$

if $[J]$ is strictly diagonal, i.e.

$$[J] = [D]$$

then each η_i in Eq. (6) has the form

$$\eta_i = A_i \, e^{-D_i t}$$

where D_i is the i^{th} diagonal term of [D].

In this case if D_i has a non negative real part each η_i is stable and so each x_i and $\dot{x}_i$ is stable and so the system is stable. Hence if the problem is reducible to strictly diagonal form in 2N space and if the roots of the system have non positive real parts the system is stable. Naturally the subclass of cases where the system is reducible to diagonal form in N space are also stable if the roots have non positive real parts.

If, however, the system is reducible only to Jordan form in 2N space, the system is stable only if the roots which are associated with the non-diagonal portion of the reduced form are not purely imaginary.

$$[I]\{\dot{Z}\} + [J]\{Z\} = 0 \tag{7}$$

If [J] is in Jordan form then there will be at least 2 equations in Eq. (7) having the form

$$\dot{\eta}_i + \lambda_i \eta_i = 0 \tag{8}$$

$$\dot{\eta}_{i-1} + \lambda_i \eta_{i-1} + \eta_i = 0 \tag{9}$$

for some i.

From Eq. (8) $\qquad \eta_i = A_i e^{-\lambda_i t}$

From Eq. (9) $\qquad \dot{\eta}_{i-1} + \lambda_i \eta_{i-1} = -A_i e^{-\lambda_i t}$

$$\therefore \quad \eta_{i-1} = B_{i-1} e^{-\lambda_i t} + (-A_i t\, e^{-\lambda_i t}) \tag{10}$$

From Eq. (10) it is seen that unless the roots of the system have negative real parts the system will be unstable. For any negative non zero real part of the root $-\lambda_i$ the term $t\,e^{-\lambda_i t}$ is bounded.

Thus if the system is strictly diagonalizable in 2N space and if the roots have non positive real parts the system is stable. However, if the problem can only be reduced to Jordan form in 2N space (these problems contain the problems that are reducible to Jordan form in N space as a sub class) the roots must still have non-positive real parts but those roots associated with the non-diagonal part of the reduced form must have non zero real parts.

Many authors have tried to develop simple criteria of stability for linear systems. However, most of the work to date is not very useful for construction or design purposes. The well known work of Routh for determining if the real parts of the roots are positive or negative is useful but rather tedious to use in any large scale system. As shown above if all the roots have negative non zero real parts the system will be stable but in practice some roots may be purely imaginary and in this case the system need not be stable.

Sylvester first showed that if [M] and [K] are symmetric and [M] is positive definite and [K] at least non negative definite the system is stable. This follows readily from the above analysis for if [M] and [K] are symmetric they are both reducible to diagonal form and as [M] is positive definite and [K] non negative definite the roots of the system are purely imaginary or zero. Likewise if [C], a symmetric and non negative definite matrix is added to this system as a damping matrix the complete system is stable.

<u>Liapunoff stability criterion</u>:[10] To extend somewhat the rather particular results of the last paragraph use is made of Liapunoff's test for stability. If a function $V(x, \dot{x}) \geq 0$ all t, can be formed of the variables of the system such that $\dot{V}(x, \dot{x}) \leq 0$ all t, then the system is stable. The difficulty

with this criterion is that one has to find $V(x, \dot{x})$ first and if one cannot find $V(x, \dot{x}) > 0$ such that $\dot{V}(x, \dot{x}) < 0$ one cannot conclude that the system is unstable. Liapunoff has in fact developed many other criteria of stability but these do not appear to have much value in the problem of the stability of general linear damped systems.

Consider the equations of a multi-degree of freedom linear system in N space

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \tag{11}$$

where $[M]$ and $[K]$ are symmetric

$[M]$ is positive definite, $[K]$ is non-negative definite

$[C]$ in general can be decomposed into a symmetric and skew symmetric part

$$[C] = [C]_{ss} + [C]_{s}$$

$[C]_{ss}$ = skew symmetric matrix

$[C]_{s}$ = symmetric matrix

Let

$$V(x, \dot{x}) = \{\dot{x}\}^T[M]\{\dot{x}\} + \Big(\{x\}^T[C]^T + \{\dot{x}\}^T[M]\Big)\Big([C]\{x\} + [M]\{\dot{x}\}\Big) + \{x\}^T[K]\{x\} \tag{12}$$

The second term of Eq. (12) is the inner product of a vector with itself and so is always non-negative. Hence if $[M]$ and $[K]$ are positive definite $V(x, \dot{x}) > 0$ for all non zero values of x and $\dot{x}$. But Eq. (12) is also a proper Liapunoff function if

$[M]$ is positive definite

$[K]$ is non-negative definite

$[C]$ is non singular

As [M] is positive definite the first term of Eq. (12) is positive for all $\dot{x} \neq 0$ and the 2nd and 3rd terms of Eq. (12) are non-negative all x, $\dot{x}$, hence $V(x, \dot{x}) > 0$ all $\dot{x} \neq 0$ all x. For $V(x, \dot{x})$ to be a proper Liapunoff function it must be greater than zero for all x, $\dot{x} \neq 0$ and for all $\dot{x}$, $x \neq 0$.

Consider the 2nd term

$$D = \left[\{x\}^T[C]^T + \{\dot{x}\}[M]^T\right]\left[[C]\{x\} + [M]\{\dot{x}\}\right]$$

$$\text{if } \{\dot{x}\} = \{0\}$$

$$D = x^T C^T C x$$

This is the inner product of the vector Cx with itself and so is zero only when

$$C x = 0 \tag{13}$$

As C is non singular $[C]^{-1}$ exists and the unique solution to Eq. 13 is $x = 0$. Therefore Eq. (12) is equal to zero only if x and $\dot{x}$ are simultaneously zero.

Hence $V(x, \dot{x}) > 0$ all $x, \dot{x} \neq 0$ if

a) [M] , [K] are both positive definite

or (14)

b) [M] positive definite, [C] non singular definite

[K] non negative definite

To show that the system is stable

$$V(x,\dot{x}) = \{\dot{x}\}^T[M]\{\dot{x}\} + \left(\{x\}^T[C]^T + \{\dot{x}\}^T[M]\right)\left([C]\{x\} + [M]\{\dot{x}\}\right) + \{x\}^T[K]\{x\}$$

$$\begin{aligned}\dot{V}(x,\dot{x}) = {} & \{\ddot{x}\}^T[M]\{\dot{x}\} + \{\dot{x}\}^T[M]\{\ddot{x}\} + \left(\{\dot{x}\}^T[C]^T + \{\ddot{x}\}^T[M]\right) \\ & \left([C]\{x\} + [M]\{\dot{x}\}\right) + \left(\{x\}^T[C]^T + \{\dot{x}\}^T[M]\right) \\ & \left([C]\{\dot{x}\} + [M]\{\ddot{x}\}\right) + \{\dot{x}\}^T[K]\{x\} + \{x\}^T[K]\{\dot{x}\} \end{aligned} \tag{15}$$

Now

$$\begin{aligned}\{\ddot{x}\}^T[M] &= -\left(\{\dot{x}\}^T[C]^T + \{x\}^T[K]\right) \\ [M]\{\ddot{x}\} &= -\left([C]\{\dot{x}\} + [K]\{x\}\right)\end{aligned} \tag{16}$$

On substituting Eq. (16) into Eq. (15)

$$\begin{aligned}\therefore\ \dot{V}(x,\dot{x}) = {} & -\{\dot{x}\}^T[C]^T\{\dot{x}\} - \{x\}^T[K]\{\dot{x}\} - \{\dot{x}\}^T[C]\{\dot{x}\} - \{\dot{x}\}^T[K]\{x\} \\ & - \{x\}^T[K][C]\{x\} - \{x\}^T[K][M]\{\dot{x}\} - \{x\}^T[C]^T[K]\{x\} \\ & - \{\dot{x}\}^T[M][K]\{x\} + \{\dot{x}\}^T[K]\{x\} + \{x\}^T[K]\{\dot{x}\} \\ = {} & -2\{\dot{x}\}^T[C]_s\{\dot{x}\} - \{x\}^T[K][C]\{x\} - \{x\}^T[C]^T[K]\{x\} \\ & - \{x\}^T[K][M]\{\dot{x}\} - \{\dot{x}\}^T[M][K]\{x\} \\ = {} & -2\{\dot{x}\}^T[C]_s\{\dot{x}\} - 2\{x\}^T[K][C]\{x\} \\ & -2\{x\}^T[K][M]\{\dot{x}\}\end{aligned} \tag{17}$$

Equation (17) may be written as

$$-\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}^T \left[\begin{array}{c|c} [K][C] + [C]^T[K] & [K]\,[M] \\ \hline [K][M] & [C] + [C]^T \end{array}\right] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

$\therefore \qquad \dot{V}(X\,\dot{X}) \le 0 \qquad x,\ \dot{x} \ne 0 \quad$ if

$$\left[\begin{array}{c|c} [K][C] + [C]^T[K] & [K][M] \\ \hline [K][M] & [C] + [C]^T \end{array}\right] \quad \text{is non-negative definite} \tag{18}$$

Therefore, if M, K are both symmetric and M is positive definite

and $\left[\begin{array}{c|c} [K][C] + [C]^T[K] & [K][M] \\ \hline [K][M] & [C] + [C]^T \end{array}\right]$ is non negative definite

the system is stable in Liapunoff's sense. Note that condition (18) includes the condition that $[C]_s$ be non negative definite.

These conditions are sufficient though not necessary. To illustrate this more vividly let the following conditions be specified.

If $[M] \equiv [I]$ the identity matrix C, K are both symmetric and at least non-negative definite Eq. (18) reduces to

$$\left[\begin{array}{c|c} [K][C] + [C][K] & [K] \\ \hline [K] & 2[C] \end{array}\right] \tag{19}$$

But as is well known there are no other requirements for stability once

$$M = I$$

C, K are both symmetric and at least non-negative definite.

But the non-negative definiteness of Eq. (19) is not guaranteed by C and K being non-negative definite.

As a numerical example

$$\text{if} \quad [C] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and } [K] = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

then both [C] and [K] are symmetric and positive definite but

$$[C][K] + [K][C] = \begin{bmatrix} 2 & -11 \\ -11 & 40 \end{bmatrix}$$

which is not non-negative definite and hence (19) is not non-negative definite.

As a final example of the use of Liapunoff's method the following theorem is proved.

Theorem: For $I\ddot{x} + C\dot{X} + Kx = 0$ (20)

$$K = K_s + K_{ss} \; ; \; C = C_s + C_{ss}$$

s = symmetric part

ss = skew symmetric part (zeros on diagonal)

if (i) K_s is positive definite

and (ii) $\left[\begin{array}{c|c} C_s & K_{ss} \\ \hline {K_{ss}}^T & (K^T C)_s \end{array}\right]$ is non negative definite

then stability is assured. This as before is a sufficient though not a necessary condition.

Proof: Define the Liapunoff function $V(x, \dot{x})$ as follows

$$V = \frac{1}{2}\dot{x}^T\dot{x} + \frac{1}{2}(x^TC^T + \dot{x}^T)(Cx + \dot{x}) + x^T(K^T + K)x$$

$\dot{x}^T\dot{x}$, $(x^TC^T + \dot{x}^T)(Cx + \dot{x})$ being inner products of vectors are always non-negative. As $K_s = \frac{1}{2}(K^T + K)$ is positive definite $V(x, \dot{x})$ is non-negative.

$$\dot{V}(x\,\dot{x}) = \frac{1}{2}\ddot{x}^T\dot{x} + \frac{1}{2}\dot{x}^T\ddot{x} + \frac{1}{2}(\dot{x}^TC^T + \ddot{x}^T)(Cx + \dot{x})$$

$$+ \frac{1}{2}(x^TC^T + \dot{x}^T)(C\dot{x} + \ddot{x}) + \dot{x}^T(K^T + K)x + x^T(K^T + K)\dot{x} \qquad (21)$$

From (20) $\ddot{x} = -(C\dot{x} + Kx)$; $\ddot{x}^T = -(\dot{x}^TC^T + x^TK^T)$

$$(\dot{x}^TC^T + \ddot{x}^T) = -(x^TK^T) \text{ and } (C\dot{x} + \ddot{x}) = -(Kx) \qquad (22)$$

On substituting Eq. (22) into (21)

$$\dot{V} = -\frac{1}{2}\Big\{\dot{x}^T(C^T + C)\dot{x} + x^T(K^TK)\dot{x} + \dot{x}^T(K - K^T)x$$

$$+ x^T(K^TC + C^TK)x\Big\} \qquad (23)$$

$$= -\left[\dot{x}^TC_s\dot{x} + x^TK_{ss}\dot{x} + \dot{x}^TK_{ss}^T + x^T(K^TC)_s x\right] \qquad (24)$$

$$= -\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix}\left[\begin{array}{c|c} C_s & K_{ss} \\ \hline K_{ss}^T & (K^TC)_s\end{array}\right]\begin{Bmatrix}\dot{x}\\ x\end{Bmatrix} \qquad (25)$$

Hence if $\left[\begin{array}{c|c} C_s & K_{ss} \\ \hline K_{ss}^T & (K^TC)_s\end{array}\right]$ is non negative definite the system is stable.

As before the matrix criterion of stability of this theorem is too restrictive if the specifications on the matrices of the system are tightened. This is merely an illustration of the well known mathematical fact that sufficient conditions on classes of results may be too restrictive for sub-classes of these results.

If K is symmetric then the matrix criterion of the above theorem reduces to

$$\left[\begin{array}{c|c} C_s & 0 \\ \hline 0 & (KC)_s \end{array}\right] \quad \text{(To be non negative definite for stability)} \qquad (26)$$

or C_s and $(KC)_s$ both be non negative definite for stability.

In the above work the product of two positive definite matrices appears quite often. Unfortunately the fact that each matrix is separately positive definite does not imply the positive definiteness of their product. Basically the reason is that the product of symmetric matrices is not necessarily symmetric, in fact it will only be symmetric if the matrices commute. If two matrices commute and are each positive definite then their product is also positive definite. This follows directly from the fact that 2 symmetric commuting matrices have the same set of eigenvectors and these eigenvectors form a complete set. For if K and C are symmetric and positive definite and they commute then

$$y^T(KC)\,y > 0 \qquad \text{for all } y \neq 0.$$

For let $y = QZ$ where Q is the eigenvector matrix of K and C.

$$\therefore\; Z^TQ^TKQQ^TCQZ = Z^T\bar{K}\bar{C}Z > 0$$
$$\text{all } Z \neq 0 \;.$$

It is interesting to note the difference between Eq. (19) and Eq. (26) realizing that $KC + CK = 2(KC)_s$.

<u>Sufficient conditions for instability of a system:</u>

For the system

$$I\ddot{x} + C\dot{x} + Kx = 0 \qquad (23)$$

if (i) C is symmetric and non-negative definite.

(ii) the symmetric part of K, K_s is negative definite then the system is unstable.

Let the Liapunoff function V be defined as

$$V = \dot{x}^T x + \frac{1}{2} x^T C x$$

$$\dot{V} = \ddot{x}^T x + \dot{x}^T \dot{x} + \frac{1}{2} \dot{x}^T C x + \frac{1}{2} x^T C \dot{x}$$

From (23) $$\ddot{x}^T = -(\dot{x}^T C + x^T K^T)$$

$$\dot{V} = -x^T K^T x + \dot{x}^T \dot{x} = -x^T K_s x + \dot{x}^T \dot{x}$$

As K_s is negative definite $\dot{V}(x, \dot{x}) > 0$ if $x \neq 0$ and $\dot{x} = 0$.

$\therefore$ if $\dot{x}^T x$ is non-negative the system is unstable as then both V and $\dot{V}$ have the same sign. Hence there are regions in the $x\ \dot{x}$ plane which are unstable.

This brief discussion of the stability of linearly damped systems indicates that a general criterion for stability is not easy to construct and that its use would be limited due to the many special cases that need to be considered.

Appendix

Example 1 — 2 Degrees of Freedom
Damping Matrix Skew Symmetric
Spring Matrix Symmetric

Equations of Motion:

$$M\ddot{x} + C\dot{X} + Kx = 0 \tag{1}$$

$$\left.\begin{aligned} \text{Let } M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ K &= \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix} \end{aligned}\right\}$$

As [C] is not symmetric it is best to transform to 2N space without trying to diagonalize problem in N space.

In 2 N space

$$[R]\{\dot{Z}\} + [K]\{Z\} = 0 \tag{2}$$

$$\begin{bmatrix} O & M \\ M & C \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} = 0 \tag{3}$$

$$Z = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} \qquad R^{-1} = \begin{bmatrix} -M^{-1}CM^{-1} & M^{-1} \\ M^{-1} & 0 \end{bmatrix}$$

Premultiply (3) by R^{-1}

$$\{\dot{Z}\} + [R^{-1}][K]\{Z\} = 0 \tag{4}$$

Expanding (4)

$$\dot{Z} + \begin{bmatrix} +M^{-1}C & M^{-1}k \\ -I & 0 \end{bmatrix} Z = 0 \tag{5}$$

$$\text{where } M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rewriting (5)

$$\{\dot{Z}\} + [P]\{Z\} = \{0\} \tag{6}$$

$$\text{where} \quad [P] = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 6 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{7}$$

To determine eigenvalues

$$|(P - \lambda I)| = \left\| \begin{bmatrix} -\lambda & -1 & 2 & -1 \\ 1 & -\lambda & -1 & 6 \\ -1 & 0 & -\lambda & 0 \\ 0 & -1 & 0 & -\lambda \end{bmatrix} \right\| = 0 \tag{8}$$

Expanding (8) beginning in the last row

$$(1)\big(\lambda(-6\lambda + 1) - 1\,(11)\big) \quad \lambda^2(\lambda^2 + 1) - \lambda\,(1 + 2\lambda) = 0 \tag{9}$$

or

$$-6\lambda^2 + \lambda - 11 - \lambda^4 - \lambda^2 - \lambda - 2\lambda^2 = 0$$

$$-\lambda^4 - 9\lambda^2 - 11 = 0 \tag{10}$$

$$\lambda^4 + 9\lambda^2 + 11 = 0$$

$$\therefore \quad \lambda^2 = \frac{-9 \pm \sqrt{37}}{2}$$

$$= \frac{-9 \pm 6.0828}{2} = \frac{-2.9172}{2} \text{ or } -\frac{15.0828}{2}$$

$$= -1.4586 \text{ or } -7.5414$$

$$\therefore \quad \lambda = \pm i(1.2077)\,, \quad \pm i\,(2.7462) \tag{11}$$

Taking the eigenvalue $\lambda_1 = +1.2077i$ to determine the associated eigenvector

$$(P - \lambda_1 I) = \begin{bmatrix} -1.2077i & -1 & 2 & -1 \\ 1 & -1.2077i & -1 & 6 \\ -1 & 0 & -1.2077i & 0 \\ 0 & -1 & 0 & -1.2077i \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix}$ is the associated eigenvector

Expanding (12)

$$\begin{aligned} b &= -1.2077i \\ a &= (-1.2077i)\,c \\ (-1-1.2077i)c &= -6 + 1.4585 \\ c &= \frac{-4.5415}{-2.4585}(1 - 1.2077i) \\ &= 1.8473(1 - 1.2077i) \\ c &= 1.8473 - 2.2310i \\ a &= -2.2310i - 2.6944 \\ &= -2.6944 - 2.2310i \end{aligned}$$

$$x_{\cdot 1} = \begin{Bmatrix} -2.6944 - 2.2310i \\ -1.2077i \\ 1.8473 - 2.2310i \\ 1 \end{Bmatrix}$$

Similarly with eigenvalue $-1.2077i$ the associated eigenvector X_2

$$x_{\cdot 2} = x_1^{*} = \begin{Bmatrix} -2.6944 + 2.2310i \\ 1.2077i \\ 1.8473 + 2.2310i \\ 1 \end{Bmatrix}$$

* denotes complex conjugate

To determine the eigenvector associated with $\lambda_2 = 2.7462\,i$

$$\begin{bmatrix} -2.7462\,i & -1 & 2 & -1 \\ 1 & -2.7462\,i & -1 & 6 \\ -1 & 0 & -2.7462\,i & 0 \\ 0 & -1 & 0 & -2.7462\,i \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} a \\ b \\ c \\ 1 \end{Bmatrix} = \text{associated eigenvector}$$

$$b = -2.7462\,i$$

$$a = (-2.7462\,i)\,c$$

$$(-1-2.7462\,i)\,c = -6 + 7.5414$$

$$c = \frac{1.5414}{8.5414}(-1 + 2.7462\,i)$$

$$= -.1805 + .4957\,i$$

$$a = 1.3613 + .4957\,i$$

$$x_3 = \begin{Bmatrix} 1.3613 + .4957\,i \\ -2.7462\,i \\ -.1805 + .4957\,i \\ 1 \end{Bmatrix}$$

Similarly x_4 the eigenvector associated with the eigenvalue $-2.7462\,i$

$$x_4 = x_3^* = \begin{Bmatrix} 1.3613 - .4957\,i \\ 2.7462\,i \\ -.1805 - 4957\,i \\ 1 \end{Bmatrix}$$

Now to prove that the original system can be diagonalized, τ the transformation matrix is constructed

$$\tau = \begin{array}{c} \begin{bmatrix} 1.3613 + .4957i & 1.3613 - .4957i & -2.6944 - 2.2310i & -2.6944 + 2.2310i \\ -2.7462i & 2.7462i & -1.2077i & 1.2077i \\ -.1805 + .4957i & -.1805 - .4957i & 1.8473 - 2.2310i & 1.8473 + 2.2310i \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccc} \qquad X_3 \qquad\qquad & \qquad X_4 \qquad\qquad & \qquad X_1 \qquad\qquad & \qquad X_2 \qquad \end{array} \end{array}$$

τ^{-1} is determined by expanding this 4 x 4 matrix into an 8 x 8 matrix by replacing each a + ib element with 4 elements $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and inverting the 8 x 8 matrix and then taking the reverse transformation to get back the complex matrix

$$\tau^{-1} = \begin{bmatrix} .0822 - .0299i & .1659i & -.0822 - .0599i & .3733 + .0299i \\ .0822 + .0299i & -.1659i & -.0822 + .0599i & .3733 - .0299i \\ -.0822 + .0681i & .0369i & .0822 + .1361i & .1267 - .0681i \\ -.0822 - .0681i & -.0369i & .0822 - .1361i & .1267 + .0681i \end{bmatrix}$$

$$\tau^{-1} P \tau = \begin{bmatrix} 2.7462i & 0 & 0 & 0 \\ 0 & -2.7462i & 0 & 0 \\ 0 & 0 & 1.2077i & 0 \\ 0 & 0 & 0 & -1.2077i \end{bmatrix}$$

This proves that P can be diagonalized by τ.

Example 2

Solution of Multi degree of freedom system with repeated roots but not sufficient eigenvectors to span the space.

System:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{X} + \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} X + \begin{bmatrix} 6 & -1 \\ -1 & 1 \end{bmatrix} \dot{X} = 0 \tag{1}$$

The system possesses non classical damping and the problem must be transformed to 2N space.

As before

$$[u] = \begin{bmatrix} 0 & I \\ -K^{-1}M & -K^{-1}C \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{4}{27} & -\frac{1}{9} & -\frac{7}{9} & \frac{1}{27} \\ -\frac{1}{9} & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{9} \end{bmatrix} \tag{3}$$

To determine the eigenvalues solve frequency equation

$$\left\| \left[[u] - \lambda I \right] \right\| = 0 \tag{4}$$

$$27\lambda^4 + 27\lambda^3 + 18\lambda^2 + 7\lambda + 1 = 0$$

$$\lambda = -\frac{1}{3}, \quad -\frac{1}{3}, \quad -\frac{1 \pm \sqrt{-11}}{6} \tag{5}$$

To determine eigenvectors:

$$\lambda = -\frac{1}{3}$$

$$\begin{bmatrix} \frac{1}{3} & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & 1 \\ -\frac{4}{27} & -\frac{1}{9} & -\frac{4}{9} & \frac{1}{27} \\ -\frac{1}{9} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{9} \end{bmatrix} \begin{Bmatrix} 1 \\ a \\ b \\ c \end{Bmatrix} = \{0\} \tag{6}$$

$$\begin{aligned} \frac{1}{3} + b &= 0 \\ \frac{1}{3}a + c &= 0 \\ -\frac{4}{27} - \frac{1}{9}a - \frac{4}{9}b + \frac{1}{27}c &= 0 \\ -\frac{1}{9} - \frac{1}{3}a - \frac{1}{3}b - \frac{1}{9}c &= 0 \end{aligned} \tag{7}$$

$$\therefore \quad \begin{Bmatrix} 1 \\ 0 \\ -\frac{1}{3} \\ 0 \end{Bmatrix} = \phi_{-\frac{1}{3}}$$

is the <u>unique</u> eigenvector corresponding to $\lambda = -\frac{1}{3}$

This eigenvector is unique as (7) has only one set of solutions.

$\lambda = -\frac{1 + \sqrt{-11}}{6}$ has associated eigenvector determined as follows:

$$\begin{bmatrix} \frac{1}{6} - \frac{\sqrt{-11}}{6} & 0 & 1 & 0 \\ 0 & \frac{1}{6} - \frac{\sqrt{-11}}{6} & 0 & 1 \\ -\frac{4}{27} & -\frac{1}{9} & -\frac{33}{4} - \frac{\sqrt{-11}}{6} & \frac{1}{27} \\ -\frac{1}{9} & -\frac{1}{3} & -\frac{1}{3} & -\frac{3}{54} - \frac{\sqrt{-11}}{54} \end{bmatrix} \begin{Bmatrix} 1 \\ a \\ b \\ c \end{Bmatrix} = 0 \qquad (8)$$

$$\phi_{-\frac{1+\sqrt{-11}}{6}} = \begin{Bmatrix} 1 \\ \frac{5}{2} - \frac{1}{2}\sqrt{-11} \\ -\frac{1}{6} + \frac{\sqrt{-11}}{6} \\ \frac{1}{2} + \frac{1}{2} \cdot \sqrt{-11} \end{Bmatrix} \qquad (9)$$

Hence

$$\phi_{\left(\frac{-1+\sqrt{-11}}{6}\right)} = \begin{Bmatrix} 1 \\ \frac{5}{2} + \frac{1}{2}\sqrt{-11} \\ -\frac{1}{6} - \frac{\sqrt{-11}}{6} \\ \frac{1}{2} - \frac{1}{2}\sqrt{-11} \end{Bmatrix} \qquad (10)$$

To determine generalized eigenvectors corresponding to

$$\lambda = -\frac{1}{3}$$

$$(L - \lambda I)\left[\lambda = -\frac{1}{3}\right] = \begin{bmatrix} \frac{1}{3} & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & 1 \\ -\frac{4}{27} & -\frac{1}{9} & -\frac{4}{9} & \frac{1}{27} \\ -\frac{1}{9} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{9} \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 9 & 0 & 27 & 0 \\ 0 & 9 & 0 & 27 \\ -4 & -3 & -12 & 1 \\ -3 & -9 & -9 & 3 \end{bmatrix}$$

$$(L - \lambda I)^2 = \frac{1}{27^2} \begin{bmatrix} -27 & -81 & -81 & 27 \\ -81 & -162 & -243 & 324 \\ 9 & 0 & 27 & -90 \\ 0 & -81 & 0 & -243 \end{bmatrix} \qquad (11)$$

As the missing eigenvector is a generalized eigenvector of rank 2

$$x = \{x_1, x_2, x_3, x_4\}$$

From Eq. (11) as there are only 2 independent rows

$$\left.\begin{aligned} x_1 + 3x_2 + 3x_3 - x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 - 4x_4 &= 0 \end{aligned}\right\} \quad \text{2 Indept. Eqn. to define } N_2 \qquad (12)$$

From Eq. (12)

$$\begin{aligned} \therefore \quad x_2 &= -3x_4 \\ x_1 &= -3x_3 + 10x_4 \end{aligned} \qquad (13)$$

$$\therefore \{x_1\} \quad \begin{Bmatrix} -3x_3 + 10x_4 \\ -3x_4 \\ x_3 \\ x_4 \end{Bmatrix} \quad \text{vector in Null space of matrix Eq. (11)} \qquad (14)$$

$$(L - \lambda I)\{x_1\} = \begin{Bmatrix} -27x_3 + 90x_4 + 27x_3 \\ -27x_4 + 27x_4 \\ +27x_3 - 40x_4 + 9x_4 - 12x_3 + x_4 \\ 9x_3 - 30x_4 + 27x_4 - 9x_3 + 3x_4 \end{Bmatrix} \qquad (15)$$

$$= \begin{Bmatrix} 90x_4 \\ 0 \\ -30x_4 \\ 0 \end{Bmatrix} = x_4 \begin{Bmatrix} 90 \\ 0 \\ -30 \\ 0 \end{Bmatrix} \qquad (16)$$

$\therefore \quad \xi_1 = \{10 \; -3 \quad 0 \quad 1\}$ is the generalized eigenvector

$\left[\{10, \; -3 \text{ needed to be in } N_2\}\right]$

Null space N_1 defined by $X_4 = 0$ and

from Eq. 13 $\therefore$ $X_2 = 0$

$X_1 = -3X_3$

$$\therefore \quad \xi_2 = \begin{Bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

Check

$$(L - \lambda I)\,\xi_1 = \begin{Bmatrix} +\frac{10}{3} \\ 0 \\ -\frac{30}{27} \\ 0 \end{Bmatrix} = -\frac{10}{9}\begin{Bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tag{17}$$

$$\text{Take } \xi_1 = \frac{3}{10}\begin{Bmatrix} 10 \\ -3 \\ 0 \\ 1 \end{Bmatrix} \tag{18}$$

$$\xi_2 = \begin{Bmatrix} +1 \\ 0 \\ -\frac{1}{3} \\ 0 \end{Bmatrix} \tag{19}$$

This method of determining the generalized eigenvectors is essentially the same as discussed in Chapter 1.

Let A be the required transformation matrix

$$\therefore \quad A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & \frac{9}{10} & \frac{5}{2} - \frac{1}{2}\sqrt{-11} & \frac{5}{2} + \frac{1}{2}\sqrt{-11} \\ -\frac{1}{3} & 0 & -\frac{1}{6} + \frac{1}{6}\sqrt{-11} & -\frac{1}{6} - \frac{1}{6}\sqrt{-11} \\ 0 & \frac{3}{10} & \frac{1}{2} + \frac{1}{2}\sqrt{-11} & \frac{1}{2} - \frac{1}{2}\sqrt{-11} \end{bmatrix}$$

On inverting

$$A^{-1} = \frac{1}{27\sqrt{11}} \begin{bmatrix} -5\sqrt{11} & -9\sqrt{11} & -96\sqrt{11} & 23\sqrt{11} \\ 10\sqrt{11} & 0 & 30\sqrt{11} & -10\sqrt{11} \\ \sqrt{11} + 4\,i & 4\tfrac{1}{2}\sqrt{11} + 4\tfrac{1}{2}i & 3\sqrt{11} + 12\,i & 3\tfrac{1}{2}\sqrt{11} - 26\tfrac{1}{2}\,i \\ \sqrt{11} - 4\,i & 4\tfrac{1}{2}\sqrt{11} - 4\tfrac{1}{2}i & 3\sqrt{11} - 12\,i & 3\tfrac{1}{2}\sqrt{11} + 26\tfrac{1}{2}i \end{bmatrix}$$

On multiplying

$$u\,A = \frac{1}{27\sqrt{11}} \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{6} + \frac{\sqrt{11}}{6}\,i & -\frac{1}{6} - \frac{\sqrt{11}}{6}\,i \\ 0 & \frac{3}{10} & \frac{1}{2} + \frac{1}{2}\sqrt{11}\,i & \frac{1}{2} - \frac{1}{2}\sqrt{11}\,i \\ \frac{1}{9} & -\frac{1}{3} & -\frac{5}{18} - \frac{1}{18}\sqrt{11}\,i & -\frac{5}{18} + \frac{1}{18}\sqrt{11}\,i \\ 0 & -\frac{1}{10} & -1 & -1 \end{bmatrix}$$

$$A^{-1}\,u\,A = \frac{1}{27\sqrt{11}}\begin{bmatrix} -9\sqrt{11} & 27\sqrt{11} & 0 & 0 \\ 0 & -9\sqrt{11} & 0 & 0 \\ 0 & 0 & -\frac{9}{2}\sqrt{11}+\frac{99}{2}\,i & 0 \\ 0 & 0 & 0 & -\frac{9}{2}\sqrt{11}-\frac{99}{2}\,i \end{bmatrix}$$

$$\therefore\ A^{-1}\,u\,A = \begin{bmatrix} -\frac{1}{3} & 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & (-\frac{1}{6}+\frac{1}{6}\sqrt{11}\,i) & 0 \\ 0 & 0 & 0 & (-\frac{1}{6}+\frac{1}{6}\sqrt{11}\,i) \end{bmatrix}$$

which is in Jordan's Form.

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